# Upper 6 Chapter 10

# Numerical Methods

## Chapter Overview

1. Locating Roots

2. Iteration

3. The Newton-Raphson Method

4. Applications to Modelling

| | | |
|---|---|---|
| 9.1 | Locate roots of $f(x) = 0$ by considering changes of sign of $f(x)$ in an interval of $x$ on which $f(x)$ is sufficiently well behaved.<br><br>Understand how change of sign methods can fail. | Students should know that sign change is appropriate for continuous functions in a small interval.<br><br>When the interval is too large sign may not change as there may be an even number of roots.<br><br>If the function is not continuous, sign may change but there may be an asymptote (not a root). |
| 9.2 | Solve equations approximately using simple iterative methods; be able to draw associated cobweb and staircase diagrams. | Understand that many mathematical problems cannot be solved analytically, but numerical methods permit solution to a required level of accuracy.<br><br>Use an iteration of the form $x_{n+1} = f(x_n)$ to find a root of the equation $x = f(x)$ and show understanding of the convergence in geometrical terms by drawing cobweb and staircase diagrams. |
| 9.3 | Solve equations using the Newton-Raphson method and other recurrence relations of the form<br><br>$x_{n+1} = g(x_n)$<br>Understand how such methods can fail. | For the Newton-Raphson method, students should understand its working in geometrical terms, so that they understand its failure near to points where the gradient is small. |

| | | |
|---|---|---|
| 9.4 | Understand and use numerical integration of functions, including the use of the trapezium rule and estimating the approximate area under a curve and limits that it must lie between. | For example, evaluate $\int_0^1 \sqrt{(2x+1)}\ \mathrm{d}x$ using the values of $\sqrt{(2x+1)}$ at $x = 0$, $0.25$, $0.5$, $0.75$ and $1$ and use a sketch on a given graph to determine whether the trapezium rule gives an over-estimate or an under-estimate. |
| 9.5 | Use numerical methods to solve problems in context. | Iterations may be suggested for the solution of equations not soluble by analytic means. |

## LOCATING ROOTS

Finding the root of a function $f(x)$ is to **solve the equation $\boldsymbol{f(x)=0}$**

However, for some functions, the ‘exact’ root is complicated and difficult to calculate …

For example:

$$\boldsymbol{x^3+2x^2-3x+4=0}$$

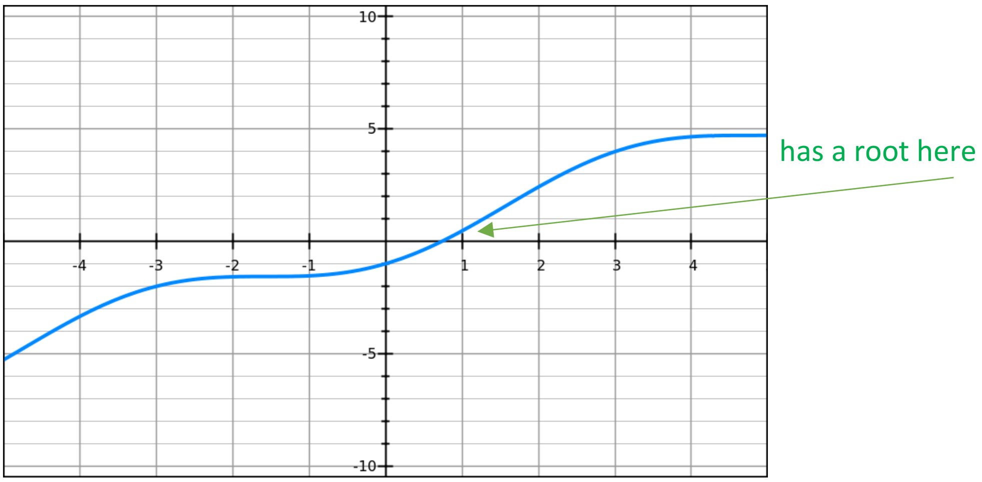
has the solution:

$$x=\frac{1}{3}\left(-2-\frac{13}{\sqrt[3]{89-6\sqrt{159}}}-\sqrt[3]{89-6\sqrt{159}}\right)$$

… or there is no ‘algebraic’ expression at all. (involving roots, logs, sin, cos, etc.)

For example:

$$\boldsymbol{x-\cos(x)=0}$$




**To show that a root exists in a given interval, show that $\boldsymbol{f(x)}$ changes sign**

**Example 1**

Show that $f(x) = e^x + 2x - 3$ has a root between $x = 0.5$ and $x = 0.6$

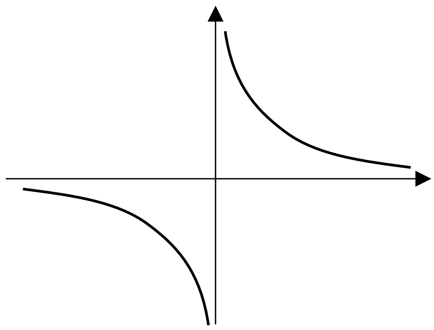
STEP 1: Find $f(x)$ for the two values given in the question

STEP 2: Write a concluding statement referring to the change in sign and the fact that $f(x)$ is a continuous function

**Note on functions that are NOT continuous:**

If the function is not continuous, the sign change may be due to an asymptote rather than a root.

For example:



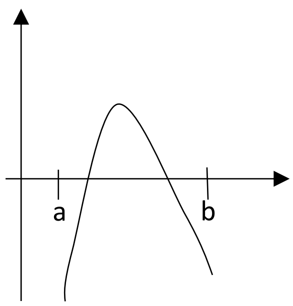
When $f(x) = \frac{1}{x}$, then $f(-1) = -1$ and $f(1) = 1$.

However, although there is a sign change, a root does not exist between $x = -1$ and $x = 1$

**Note on continuous functions:**

A continuous function could simply have an even number of roots in a given interval rather than no roots.

For example:




Here $f(a)$ is negative and $f(b)$ is also negative

However, although there are two roots, a sign change does not occur.

**Example 2** Edexcel C3 Jan 2013

$$g(x) = e^{x-1} + x - 6$$

The root of $g(x) = 0$ is $\alpha$.

(*c*) By choosing a suitable interval, show that $\alpha = 2.307$ correct to 3 decimal places.

**(3)**

**Example 3**

(a) Using the same axes, sketch the graphs of $y = \ln x$ and $y = \frac{1}{x}$. Explain how your diagrams shows that the function $y = \ln(x) - \frac{1}{x}$ has only one root.

(b) Show that this root lies in the interval $1.7 < x < 1.8$

(c) Given that the root of $f(x)$ is $\alpha$, show that $\alpha = 1.763$ correct to 3 decimal places.

Exercise 10A Page 276

# ITERATION

**To solve $f(x) = 0$ by an iterative method, rearrange into a form $x = g(x)$ and use the iterative formula $x_{n+1} = g(x_n)$**

**Example 1** Edexcel C3 Jan 2013

$$g(x) = e^{x-1} + x - 6$$

(*a*) Show that the equation $g(x) = 0$ can be written as

$$x = \ln(6 - x) + 1, \qquad x < 6.$$

**(2)**

The root of $g(x) = 0$ is $\alpha$.

The iterative formula

$$x_{n+1} = \ln(6 - x_n) + 1, \qquad x_0 = 2.$$

is used to find an approximate value for $\alpha$.

(*b*) Calculate the values of $x_1$, $x_2$ and $x_3$ to 4 decimal places.

**(3)**

(*c*) By choosing a suitable interval, show that $\alpha = 2.307$ correct to 3 decimal places.

**(3)**

a)

b) **$x_1, x_2, x_3$ represent successively better approximations of the root**

Initially type $x_0$ (i.e. 2) onto your calculator.
Now just type: $\ln(6 - ANS) + 1$
And then press your $=$ key to get successive iterations.

c)

**The starting value $x_0$matters.**

- If there are a multiple roots, the iteration might converge to (i.e. approach) a different root.
- The iteration not converge to a root at all and **diverges** (i.e. approach infinity).

**Example 2**

$$f(x) = x^3 - 3x^2 - 2x + 5$$

(a) Show that the equation $f(x) = 0$ has a root in the interval $3 < x < 4$.

(b) Use the iterative formula $x_{n+1} = \sqrt{\frac{x_n^3 - 2x_n + 5}{3}}$ to calculate the values of $x_1$, $x_2$ and $x_3$, giving your answers to 4 decimal places, and taking:
(i) $x_0 = 1.5$ (ii) $x_0 = 4$

**Staircase and cobweb diagrams**

**Example 3**

$$f(x) = x^2 - 8x + 4$$

(a) Show that the root of the equation $f(x) = 0$ can be written as $x = \sqrt{8x - 4}$

(b) Using the iterative formula $x_{n+1} = \sqrt{8x_n - 4}$, and starting with $x_0 = 1$, draw a staircase diagram, indicating $x_0, x_1,\ x_2$ on your $x$-axis, as well as the root $\alpha$.

Exercise 10B Page 280

# THE NEWTON-RAPHSON METHOD

**The Newton- Raphson method can be used to find numerical solutions to equations of the form $\boldsymbol{f(x) = 0}$. You need to be able to differentiate $\boldsymbol{f(x)}$ in order to use this method.**

**The Newton- Raphson formula is:**

$$\boldsymbol{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

**Example 1**

Recall that in lesson 1 we saw that the function $\boldsymbol{f(x) = x - \cos(x)}$ has a root, $\alpha$, in the interval $0 < \alpha < 1$.

Using $x_0 = 0.5$ as a first approximation to $\alpha$, apply the Newton-Raphson procedure three times to find a better approximation to $\alpha$ which, in this case, will be accurate to 7 decimal places.

To perform iterations quickly, do the following on your calculator:

[0.5] [=]

[ANS] – (ANS – cos(ANS))/(1 + sin(ANS))

Then press [=].

**Example 2** Edexcel FP1 June 2013(R) Q3c

$$f(x) = \frac{1}{2}x^4 - x^3 + x - 3$$

...

The equation $f(x) = 0$ has a root $\beta$ in the interval $[-2, -1]$.

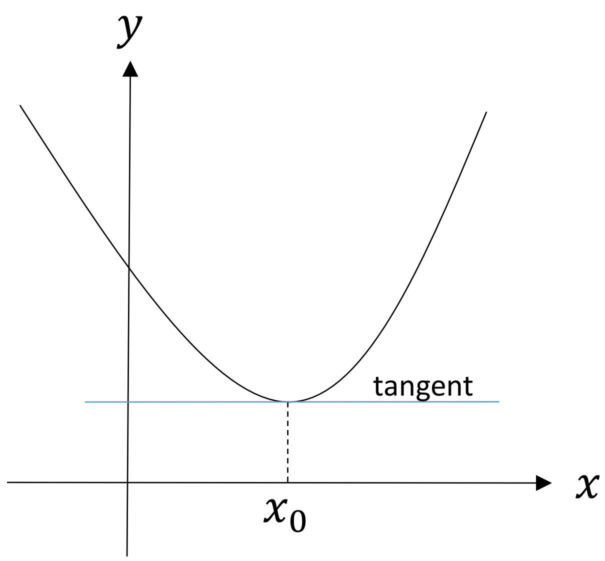
(*c*) Taking $-1.5$ as a first approximation to $\beta$, apply the Newton-Raphson process once to $f(x)$ to obtain a second approximation to $\beta$.
Give your answer to 2 decimal places.
**(5)**

**Example 3** Edexcel FP1 Jan 2010 Q2c

$$f(x) = 3x^2 - \frac{11}{x^2}.$$

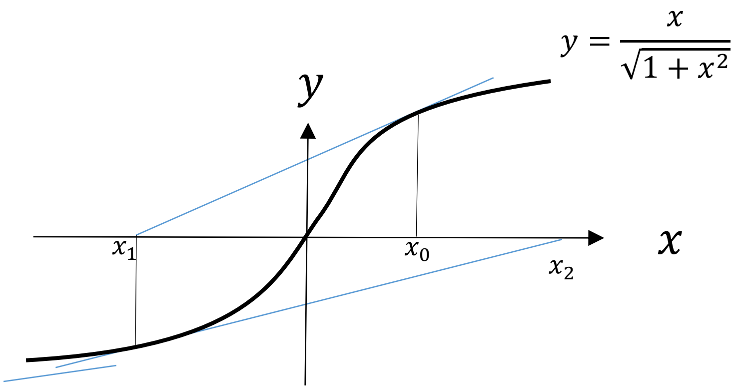
(*c*) Taking 1.4 as a first approximation to $\alpha$, apply the Newton-Raphson procedure once to $f(x)$ to obtain a second approximation to $\alpha$, giving your answer to 3 decimal places.
**(5)**

**When does Newton-Raphson fail?**




$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

**If the starting value $x_0$ was the stationary point**, then $f'(x_0) = 0$, resulting in a division by 0 in the above formula.
Graphically, it is because the tangent will never reach the $x$-axis.




Newton-Raphson also suffers from the same drawbacks as solving by iteration, in that it is possible for the values of $x_i$ to **diverge**.
In this example, the $x_i$ oscillate either side of 0, but get gradually further away from $\alpha = 0$.

Exercise 10C Page 284

## APPLICATIONS TO MODELLING

**Example 4**

The price of a car in £s, $x$ years after purchase, is modelled by the function

$$f(x) = 15\,000\,(0.85)^x - 1000 \sin x\,, \quad x > 0$$

(a) Use the model to find the value, to the nearest hundred £s, of the car 10 years after purchase.

(b) Show that $f(x)$ has a root between 19 and 20.

(c) Find $f'(x)$

(d) Taking 19.5 as a first approximation, apply the Newton-Raphson method once to $f(x)$ to obtain a second approximation for the time when the value of the car is zero. Give your answer to 3 decimal places.

(e) Criticise this model with respect to the value of the car as it gets older.

Exercise 10D Page 287