U6 Chapter 1

Algebraic Methods

Chapter Overview

1. Proof by contradiction

2. Algebraic fractions

3. Partial fractions

4. Algebraic division

Course specification

	Disproof by counter example Proof by contradiction (including proof of the irrationality of √2 and the infinity of primes, and application to unfamiliar proofs).	**Disproof by counter example** **e.g. show that the statement "$n^2 - n + 1$ is a prime number for all values of n" is untrue**
2.6	**Manipulate polynomials algebraically, including expanding brackets and collecting like terms, factorisation and simple algebraic division; use of the factor theorem.** Simplify rational expressions, including by factorising and cancelling, and algebraic division (by linear expressions only).	**Only division by $(ax + b)$ or $(ax - b)$ will be required. Students should know that if $f(x) = 0$ when $x = a$, then $(x - a)$ is a factor of $f(x)$.** **Students may be required to factorise cubic expressions such as $x^3 + 3x^2 - 4$ and $6x^3 + 11x^2 - x - 6$.** Denominators of rational expressions will be linear or quadratic, e.g. $\frac{1}{ax+b}$, $\frac{ax+b}{px^2+qx+r}$, $\frac{x^3+a^3}{x^2-a^2}$
2.10	Decompose rational functions into partial fractions (denominators not more complicated than squared linear terms and with no more than 3 terms, numerators constant or linear).	Partial fractions to include denominators such as $(ax + b)(cx + d)(ex + f)$ and $(ax + b)(cx + d)^2$. Applications to integration, differentiation and series expansions.

1 :: Proof By Contradiction

✏ To prove a statement is true by contradiction:
- **Assume** that the statement is in fact **false.**
- Prove that this would **lead to a contradiction**.
- Therefore we were wrong in assuming the statement was false, and therefore it must be true.

Prove that there is no greatest odd integer.

Assume that there is a greatest odd integer, n.

Then $n+2$ is an odd integer which is larger than n.
This contradicts the assumption that n is the greatest odd integer.

Therefore, there is no greatest odd integer.

How to structure/word proof:
1. "Assume that *[negation of statement]*."
2. [*Reasoning followed by…*] *"This contradicts the assumption that…"* or *"This is a contradiction"*.
3. "Therefore [*restate original statement*]."

Negating the original statement

The first part of a proof by contradiction requires you to negate the original statement. What is the negation of each of these statements? ***(Click to choose)***

"There are infinitely many prime numbers."

"All Popes are Catholic."

"If it is raining, my garden is wet."

More Examples

Prove by contradiction that if n^2 is even, then n must be even.

Prove by contradiction that $\sqrt{2}$ is an irrational number.

Prove by contradiction that there are infinitely many prime numbers.

Multiplying/Dividing Algebraic Fractions

As your saw at GCSE level, multiplying algebraic fractions is no different to multiply numeric fractions.
You may however need to **cancel common factors, by factorising where possible**.

$$\frac{a}{b} \times \frac{c}{a} =$$

$$\frac{x+1}{2} \times \frac{3}{x^2-1} =$$

To divide by a fraction, **multiply by the reciprocal of the second fraction**.

$$\frac{p}{q} \div \frac{r}{q} =$$

$$\frac{x+2}{x+4} \div \frac{3x+6}{x^2-16} =$$

Test Your Understanding

$$\frac{x+3}{5} \times \frac{10}{x^2-9} =$$

$$\frac{x^2+x}{y} \div \frac{x^2-x-2}{y^2} =$$

Common student "Crime against Mathematics":

$$\frac{x^2+\cancel{y}}{2\cancel{y}} = \frac{x^2}{2}$$

Adding/Subtracting Algebraic Fractions

To add/subtract two fractions, find a common denominator.

This can be achieved by multiplying the denominators and cross-multiplying the numerators:

$$\frac{3}{x+1} - \frac{2}{x+2} =$$

However, often we should see if first if there are common factors in the denominator **to avoid multiplying unnecessarily**:

$$\frac{3}{x+1} - \frac{4x}{x^2-1}$$

Test Your Understanding

[Edexcel C3 June 2013(R) Q1]

Express

$$\frac{3x+5}{x^2+x-12} - \frac{2}{x-3}$$

as a single fraction in its simplest form.

Express the following as a single fraction, giving your answer in its simplest form.

$$\frac{10x+4}{3x^2+4x+1} - \frac{3}{x+1}$$

Partial Fractions

If the **denominator is a product of a linear terms**, it can be split into the sum of 'partial fractions', where **each denominator is a single linear term**.

$$\frac{6x-2}{(x-3)(x+1)} \equiv \frac{A}{x-3} + \frac{B}{x+1}$$

Notation reminder: $\equiv$ means 'equivalent/identical to', and indicates that both sides are equal for **all** values of x.

Further Example

Given that $\frac{6x^2+5x-2}{x(x-1)(2x+1)} \equiv \frac{A}{x} + \frac{B}{x-1} + \frac{C}{2x+1}$, find the values of the constants A, B, C.

Test Your Understanding

C4 June 2005 Q3a

Express $\dfrac{5x+3}{(2x-3)(x+2)}$ in partial fractions.

(3)

Repeated linear factors

Suppose we wished to express $\frac{2x+1}{(x+1)^2}$ as $\frac{A}{x+1} + \frac{B}{x+1}$. What's the problem?

Q Split $\frac{11x^2+14x+5}{(x+1)^2(2x+1)}$ into partial fractions.

$$\frac{11x^2+14x+5}{(x+1)^2(2x+1)} \equiv \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{2x+1}$$

The problem is resolved by having the factor **both squared and non-squared.**

Test Your Understanding

C4 June 2011 Q1

$$\frac{9x^2}{(x-1)^2(2x+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(2x+1)}.$$

Find the values of the constants A, B and C. **(4)**

Dealing with Improper Fractions

In Pure Year 1, we saw that the '**degree**' of a polynomial is the highest power, e.g. a quadratic has degree 2.

An algebraic fraction is **improper** if the degree of the numerator is **at least** the degree of the denominator.

$$\frac{x^2-3}{x+2} \qquad \frac{x+1}{x-1} \qquad \frac{x^3-x^2+3}{x^2-x}$$

A partial fraction is still improper if the degree is the same top and bottom.

Reducing to Quotient and Remainder

You know for example that as $7 \div 3 = 2\ rem\ 1$, we could write:

$$\frac{7}{3} = 2 + \frac{1}{3}$$

Similarly in general:

$$\frac{F(x)}{divisor} = Q(x) + \frac{remainder}{divisor}$$

Quotient

If $\frac{x^2+5x-9}{x+2} = Ax + B + \frac{C}{x+2}$, determine the values of A, B and C.

Test Your Understanding

Edexcel C4 June 2013 Q1

Given that

$$\frac{3x^4 - 2x^3 - 5x^2 - 4}{x^2 - 4} \equiv ax^2 + bx + c + \frac{dx + e}{x^2 - 4}, \quad x \neq \pm 2$$

find the values of the constants a, b, c, d and e.

(4)

Fro Tip: There's a missing x term in the numerator and missing x term in the denominator. Use $+0x$ to avoid gaps.

Dealing with Improper Fractions

Q Split $\frac{3x^2-3x-2}{(x-1)(x-2)}$ into partial fractions.

Test Your Understanding

C4 Jan 2013 Q3

Express $\dfrac{9x^2+20x-10}{(x+2)(3x-1)}$ in partial fractions.

(4)