Angles Between Straight Lines

To find angle between two lines…………………………………………………………………..

Example

[Jan 2008 Q6] 6. The points A and B have position vectors $2\mathbf{i} + 6\mathbf{j} - \mathbf{k}$ and $3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$ respectively.

The line l_1 passes through the points A and B.

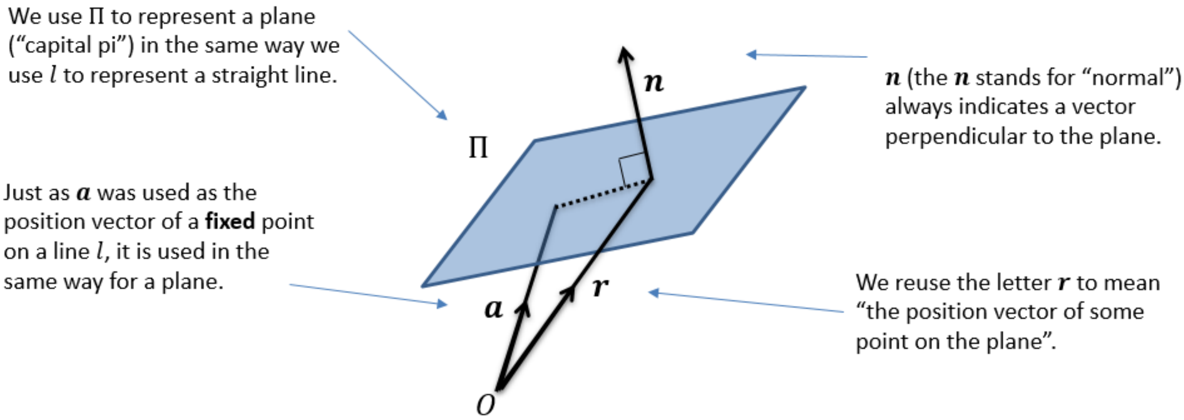
(a) Find the vector $\overrightarrow{AB}$. **(2)**

(b) Find a vector equation for the line l_1. **(2)**

A second line l_2 passes through the origin and is parallel to the vector $\mathbf{i} + \mathbf{k}$. The line l_1 meets the line l_2 at the point C.

(c) Find the acute angle between l_1 and l_2. **(3)**

Vector Equations of Planes



It's important to realise here that $\boldsymbol{n}$ and $\boldsymbol{a}$ are **fixed** for a given plane (i.e. are constant vectors), whereas $\boldsymbol{r}$ can **vary** as it represents all the possible points on the plane**.**

How could we use the dot product to find some relationship between $\boldsymbol{a}, \boldsymbol{r}, \boldsymbol{n}$?

$\boldsymbol{r} - \boldsymbol{a}$ lies on the plane and is therefore perpendicular to $\boldsymbol{n}$, thus $(\boldsymbol{r} - \boldsymbol{a}) \cdot \boldsymbol{n} = 0$

$$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$$

But since $\boldsymbol{a} \cdot \boldsymbol{n}$ is a constant, replace with constant scalar p:

Hence, the equation of plane is given by:

$$\boldsymbol{r} \cdot \boldsymbol{n} = p$$

where $\boldsymbol{r}$ is position vector of some point on the plane, $\boldsymbol{n}$ is normal to plane, $p = \boldsymbol{a} \cdot \boldsymbol{n}$ is a scalar constant.

If $\boldsymbol{r} \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = p$ is the scalar product equation of a plane, then the Cartesian form is:

$$n_1 x + n_2 y + n_3 z - p = 0$$

Example

A point with position vector $2\boldsymbol{i} + 3\boldsymbol{j} - 5\boldsymbol{k}$ lies on the plane and the vector $3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$ is perpendicular to the plane. Find the equation of the plane in:

a) Scalar product form.

b) Cartesian form.

The Angle Between a Line and a Plane

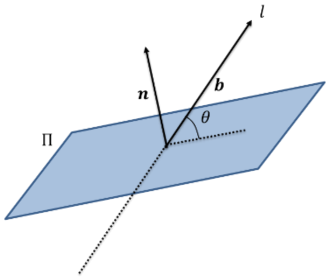
We can use the scalar product of the normal and the direction of a line to find the angle between the line and a plane.

Angle θ between line $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$ and plane $\boldsymbol{r} \cdot \boldsymbol{n} = k$:

$$\cos\alpha = \frac{\boldsymbol{b} \cdot \boldsymbol{n}}{|\boldsymbol{b}||\boldsymbol{n}|} \qquad \theta = 90° - \alpha$$

Example

Find the acute angle between the line l with equation
$\boldsymbol{r} = 2\boldsymbol{i} + \boldsymbol{j} - 5\boldsymbol{k} + \lambda(3\boldsymbol{i} + 4\boldsymbol{j} - 12\boldsymbol{k})$ and the plane with equation $\boldsymbol{r} \cdot (2\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}) = 2$.



Test Your Understanding

[June 2011 Q6] The plane P has equation

$$\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$$

(*a*) Find a vector perpendicular to the plane P.

(2)

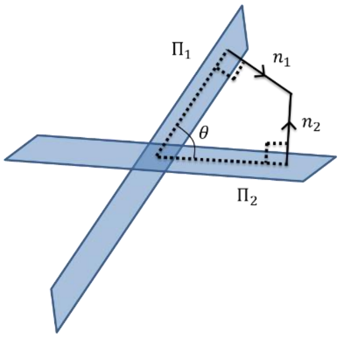
The line l passes through the point A (1, 3, 3) and meets P at (3, 1, 2).

The acute angle between the plane P and the line l is α.

(*b*) Find α to the nearest degree.

(4)

The Angle Between Two Planes



The diagram above shows why the angle between two planes is the complementary angle of the angle between the normal of both planes.

Example

Find the **acute angle** between the planes:

$$\Pi_1\text{:}\ \ \boldsymbol{r}\cdot(4\boldsymbol{i}+4\boldsymbol{j}-7\boldsymbol{k})=13 \quad \Pi_2\text{:}\ \ \boldsymbol{r}\cdot(7\boldsymbol{i}-4\boldsymbol{j}+4\boldsymbol{k})=6$$

Test Your Understanding

Find the acute angle between the planes:

$$\Pi_1: \ \boldsymbol{r} \cdot (3\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}) = 4 \qquad \Pi_2: \ \boldsymbol{r} \cdot (2\boldsymbol{i} + 3\boldsymbol{j}) = 7$$

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