Upper 6 Chapter 10

Numerical Methods

Chapter Overview

1. Locating Roots

2. Iteration

3. The Newton-Raphson Method

4. Applications to Modelling

9.1	Locate roots of $f(x) = 0$ by considering changes of sign of $f(x)$ in an interval of x on which $f(x)$ is sufficiently well behaved. Understand how change of sign methods can fail.	Students should know that sign change is appropriate for continuous functions in a small interval. When the interval is too large sign may not change as there may be an even number of roots. If the function is not continuous, sign may change but there may be an asymptote (not a root).
9.2	Solve equations approximately using simple iterative methods; be able to draw associated cobweb and staircase diagrams.	Understand that many mathematical problems cannot be solved analytically, but numerical methods permit solution to a required level of accuracy. Use an iteration of the form $x_{n+1} = f(x_n)$ to find a root of the equation $x = f(x)$ and show understanding of the convergence in geometrical terms by drawing cobweb and staircase diagrams.
9.3	Solve equations using the Newton-Raphson method and other recurrence relations of the form $x_{n+1} = g(x_n)$ Understand how such methods can fail.	For the Newton-Raphson method, students should understand its working in geometrical terms, so that they understand its failure near to points where the gradient is small.

9.4	Understand and use numerical integration of functions, including the use of the trapezium rule and estimating the approximate area under a curve and limits that it must lie between.	For example, evaluate $\int_0^1 \sqrt{(2x+1)}\ \mathrm{d}x$ using the values of $\sqrt{(2x+1)}$ at $x = 0$, 0.25, 0.5, 0.75 and 1 and use a sketch on a given graph to determine whether the trapezium rule gives an over-estimate or an under-estimate.
9.5	Use numerical methods to solve problems in context.	Iterations may be suggested for the solution of equations not soluble by analytic means.

LOCATING ROOTS

Finding the root of a function $f(x)$ is to **solve the equation $f(x) = 0$**

However, for some functions, the 'exact' root is complicated and difficult to calculate …

For example:

$$\boldsymbol{x^3 + 2x^2 - 3x + 4 = 0}$$

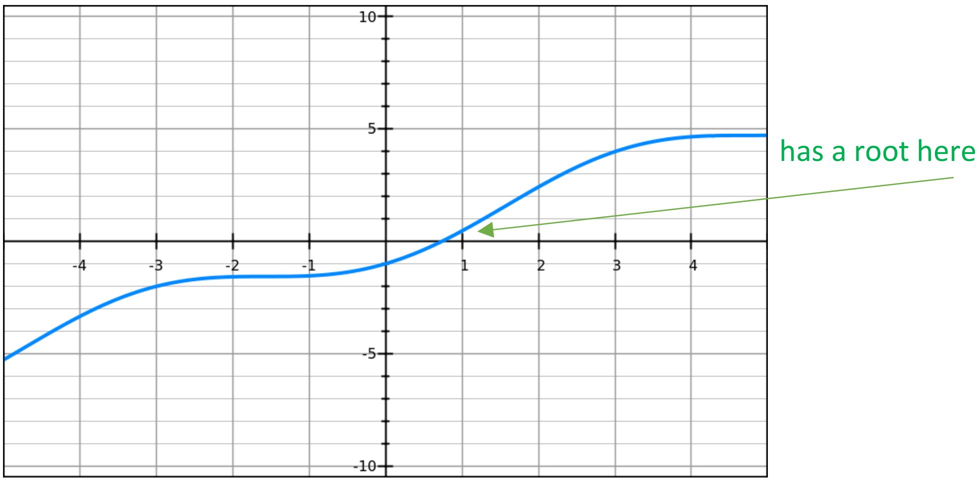
has the solution:

$$x = \frac{1}{3}\left(-2 - \frac{13}{\sqrt[3]{89 - 6\sqrt{159}}} - \sqrt[3]{89 - 6\sqrt{159}}\right)$$

… or there is no 'algebraic' expression at all. (involving roots, logs, sin, cos, etc.)

For example:

$$\boldsymbol{x - \cos(x) = 0}$$



To show that a root exists in a given interval, show that $f(x)$ changes sign

Example 1

Show that $f(x) = e^x + 2x - 3$ has a root between $x = 0.5$ and $x = 0.6$

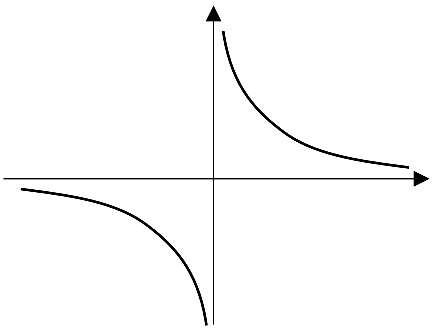
STEP 1: Find $f(x)$ for the two values given in the question

STEP 2: Write a concluding statement referring to the change in sign and the fact that $f(x)$ is a continuous function

Note on functions that are NOT continuous:

If the function is not continuous, the sign change may be due to an asymptote rather than a root.

For example:



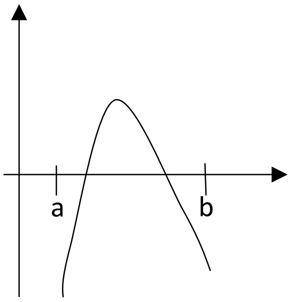
When $f(x) = \frac{1}{x}$, then $f(-1) = -1$ and $f(1) = 1$.

However, although there is a sign change, a root does not exist between $x = -1$ and $x = 1$

Note on continuous functions:

A continuous function could simply have an even number of roots in a given interval rather than no roots.

For example:



Here $f(a)$ is negative and $f(b)$ is also negative

However, although there are two roots, a sign change does not occur.

Example 2 Edexcel C3 Jan 2013

$$g(x) = e^{x-1} + x - 6$$

The root of $g(x) = 0$ is α.

(c) By choosing a suitable interval, show that $\alpha = 2.307$ correct to 3 decimal places. **(3)**

Example 3

(a) Using the same axes, sketch the graphs of $y = \ln x$ and $y = \frac{1}{x}$. Explain how your diagrams shows that the function $y = \ln(x) - \frac{1}{x}$ has only one root.

(b) Show that this root lies in the interval $1.7 < x < 1.8$

(c) Given that the root of $f(x)$ is α, show that $\alpha = 1.763$ correct to 3 decimal places.

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