Lower 6 Chapter 14

Exponentials and logarithms

Chapter Overview

1. Sketch exponential graphs.
2. Use and interpret models that use exponential functions.
3. Be able to differentiate e^{kx}.
4. Understand the log function and use laws of logs.
5. Use logarithms to estimate values of constants in non-linear models.

6 **Exponentials and logarithms**	6.1	Know and use the function a^x and its graph, where a is positive. Know and use the function e^x and its graph	Understand the difference in shape between $a < 1$ and $a > 1$
	6.2	Know that the gradient of e^{kx} is equal to ke^{kx} and hence understand why the exponential model is suitable in many applications.	Realise that when the rate of change is proportional to the y value, an exponential model should be used.
6 **Exponentials and logarithms** *continued*	6.3	Know and use the definition of $\log_a x$ as the inverse of a^x, where a is positive and $x \geqslant 0$ Know and use the function $\ln x$ and its graph Know and use $\ln x$ as the inverse function of e^x	$a \neq 1$ Solution of equations of the form $e^{ax+b} = p$ and $\ln(ax + b) = q$ is expected.
	6.4	Understand and use the laws of logarithms: $\log_a x + \log_a y = \log_a(xy)$ $\log_a x - \log_a y = \log_a\left(\frac{x}{y}\right)$ $k\log_a x = \log_a x^k$ (including, for example, $k = -1$ and $k = -\frac{1}{2}$)	Includes $\log_a a = 1$
	6.5	Solve equations of the form $a^x = b$	Students may use the change of base formula. Questions may be of the form, for example, $2^{3x-1} = 3$
	6.6	Use logarithmic graphs to estimate parameters in relationships of the form $y = ax^n$ and $y = kb^x$, given data for x and y	Plot $\log y$ against $\log x$ and obtain a straight line where the intercept is $\log a$ and the gradient is n Plot $\log y$ against x and obtain a straight line where the intercept is $\log k$ and the gradient is $\log b$
	6.7	Understand and use exponential growth and decay; use in modelling (examples may include the use of e in continuous compound interest, radioactive decay, drug concentration decay, exponential growth as a model for population growth); consideration of limitations and refinements of exponential models.	Students may be asked to find the constants used in a model. They need to be familiar with terms such as initial, meaning when $t = 0$. They may need to explore the behaviour for large values of t or to consider whether the range of values predicted is appropriate. Consideration of a second improved model may be required.

<u>Contrasting exponential graphs</u>

On the same axes sketch $y = 3^x$, $y = 2^x$, $y = 1.5^x$

On the same axes sketch $y = 2^x$ and $y = \left(\frac{1}{2}\right)^x$

<u>Graph Transformations</u>

Sketch $y = 2^{x+3}$

Exercise 14A Pg 313-314

Differentiating $\boldsymbol{y = ae^{kx}}$

If $y = e^{kx}$, where k is a constant, then $\frac{dy}{dx} = ke^{kx}$

Different e^{5x} with respect to x.

Different e^{-x} with respect to x.

Different $4e^{3x}$ with respect to x.

More Graph Transformations

Sketch $y = e^{3x}$

Sketch $y = 5e^{-x}$

Sketch $y = 2 + e^{\frac{1}{3}x}$

Sketch $y = e^{-2x} - 1$

Exercise 14B Pg 316-317

Exponential Modelling

There are two key features of exponential functions which make them suitable for **population growth**:

1. $\boldsymbol{a^x}$ **gets** $\boldsymbol{a}$ **times bigger each time** $\boldsymbol{x}$ **increases by 1. (Because** $\boldsymbol{a^{x+1} = a \times a^x}$**)**
   With population growth, we typically have a fixed percentage increase each year. So suppose the growth was 10% a year, and we used the equivalent decimal multiplier, 1.1, as a. Then 1.1^t, where t is the number of years, would get 1.1 times bigger each year.
2. **The rate of increase is proportional to the size of the population at a given moment.**
   This makes sense: The 10% increase of a population will be twice as large if the population itself is twice as large.

Example

[Textbook] The density of a pesticide in a given section of field, P mg/m^2, can be modelled by the equation $P = 160e^{-0.006t}$
where t is the time in days since the pesticide was first applied.
a. Use this model to estimate the density of pesticide after 15 days.
b. Interpret the meaning of the value 160 in this model.
c. Show that $\frac{dP}{dt} = kP$, where k is a constant, and state the value of k.
d. Interpret the significance of the sign of your answer in part (c).
e. Sketch the graph of P against t.

Exercise 14C Pg 318-319

Logarithms

$\log_a n$ ("said log base a of n") is equivalent to $a^x = n$.
The log function outputs the **missing power**.

Examples

$$\log_5 25 =$$

$$\log_3 81 =$$

$$\log_2 32 =$$

$$\log_{10} 1000 =$$

$$\log_4 1 =$$

$$\log_4 4 =$$

$$\log_2 \left(\frac{1}{2}\right) =$$

$$\log_3 \left(\frac{1}{27}\right) =$$

$$\log_2 \left(\frac{1}{16}\right) =$$

$$\log_a (a^3) =$$

$$\log_4 (-1) =$$

With your calculator…

$\log_{\square} \square$

$\log_3 7 =$
$\log_5 0.3 =$

ln

$\ln 10 =$
$\ln e =$

log

$\log 100 =$

Exercise 14D Pg 320-321

Extension

1 [MAT 2015 1J] Which is the largest of the following numbers?

A) $\frac{\sqrt{7}}{2}$ B) $\frac{5}{4}$ C) $\frac{\sqrt{10!}}{3(6!)}$

D) $\frac{\log_2 30}{\log_3 85}$ E) $\frac{1+\sqrt{6}}{3}$

2 *[MAT 2013 1F]* Three *positive* numbers a, b, c satisfy

$$\log_b a = 2$$
$$\log_b(c-3) = 3$$
$$\log_a(c+5) = 2$$

This information:

A) specifies a uniquely;
B) is satisfied by two values of a;
C) is satisfied by infinitely many values of a;
D) is contradictory

Laws of logs

Three main laws:

$$\log_a x + \log_a y = \log_a xy$$
$$\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$$
$$\log_a(x^k) = k \log_a x$$

Special cases:

$$\log_a a = 1 \quad (a > 0,\ a \neq 1)$$
$$\log_a 1 = 0 \quad (a > 0,\ a \neq 1)$$
$$\log\left(\frac{1}{x}\right) = \log(x^{-1}) = -\log(x)$$

Not in syllabus (but in MAT/PAT):

$$\log_a b = \frac{\log_c b}{\log_c a}$$

Examples

Write as a single logarithm:

a. $\log_3 6 + \log_3 7$

b. $\log_2 15 - \log_2 3$

c. $2\log_5 3 + 3\log_5 2$

d. $\log_{10} 3 - 4\log_{10}\left(\frac{1}{2}\right)$

Write in terms of $\log_a x$, $\log_a y$ and $\log_a z$

a. $\log_a(x^2yz^3)$

b. $\log_a\left(\frac{x}{y^3}\right)$

c. $\log_a\left(\frac{x\sqrt{y}}{z}\right)$

d. $\log_a\left(\frac{x}{a^4}\right)$

Solving equations with logs

Solve the equation $\log_{10} 4 + 2\log_{10} x = 2$

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Given that $2\log_2(x + 15) - \log_2 x = 6$,

(*a*) show that $x^2 - 34x + 225 = 0$.

(5)

(*b*) Hence, or otherwise, solve the equation $2\log_2(x + 15) - \log_2 x = 6$.

(2)

Exercise 14E Pg 323-324

Extension

1 *[AEA 2010 Q1b]* Solve the equation

$$\log_3(x-7) - \frac{1}{2}\log_3 x = 1 - \log_3 2$$

2 *[AEA 2008 Q5i]* Anna, who is confused about the rules of logarithms, states that

$$(\log_3 p)^2 = \log_3(p^2)$$
$$\log_3(p+q) = \log_3 p + \log_3 q$$

However, there is a value for p and a value for q for which both statements are correct. Find their values.

3 *[MAT 2007 1I]* Given that a and b are positive and

$$4(\log_{10} a)^2 + (\log_{10} b)^2 = 1$$

what is the greatest possible value of a?

4 *[MAT 2002 1F]* Observe that $2^3 = 8$, $2^5 = 32$, $3^2 = 9$ and $3^3 = 27$. From these facts, we can deduce that $\log_2 3$ is:

A) between $1\frac{1}{3}$ and $1\frac{1}{2}$

B) between $1\frac{1}{2}$ and $1\frac{2}{3}$

C) between $1\frac{2}{3}$ and 2

D) none of the above

Solving equations with exponential terms

Solve $3^x = 20$

Solve $5^{4x-1} = 61$

Solve $3^x = 2^{x+1}$

Solve the equation $5^{2x} - 12(5^x) + 20 = 0$, giving your answer to 3sf.

Solve $3^{2x-1} = 5$, giving your answer to 3dp.

Solve $2^x 3^{x+1} = 5$, giving your answer in exact form.

Solve $3^{x+1} = 4^{x-1}$, giving your answer to 3dp.

Exercise 14F Pg 325

Extension

1 [MAT 2011 1H] How many *positive* values x which satisfy the equation:

$$x = 8^{\log_2 x} - 9^{\log_3 x} - 4^{\log_2 x} + \log_{0.5} 0.25$$

2 [MAT 2013 1J] For a real number x we denote by $[x]$ the largest integer less than or equal to x. Let n be a natural number. The integral

$$\int_0^n [2^x]\ dx$$

equals:

(A) $\log_2\big((2^n - 1)!\big)$

(B) $n\,2^n - \log_2\big((2^n)!\big)$

(C) $n\,2^n$

(D) $\log_2\big((2^n)!\big)$

Natural logarithms

The inverse of $y = e^x$ is $y = \ln x$

$$\ln e^x =$$

$$e^{\ln x} =$$

Solve $e^x = 5$

Solve $2 \ln x + 1 = 5$

Solve $e^{2x} + 2e^x - 15 = 0$

Solve $e^x - 2e^{-x} = 1$

Solve $\ln(3x + 1) = 2$

Solve $e^{2x} + 5e^{x} = 6$

Solve $2^{x}e^{x+1} = 3$ giving your answer as an exact value.

Exercise 14G Pg 327-8

Graphs for Exponential Data

Turning non-linear graphs into linear ones

Case 1: Polynomial → Linear

Suppose our original model was a polynomial one*:

$$y = ax^n$$

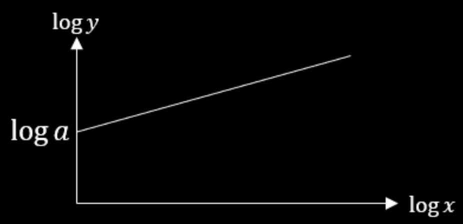
Then taking logs of both sides:

$$\log y = \log ax^n$$
$$\log y = \log a + n \log x$$

We can compare this against a straight line:

$$Y = mX + c$$

✏ If $y = ax^n$, then the graph of $\log y$ against $\log x$ will be a straight line wih gradient n and vertical intercept $\log a$.



* We could also allow non-integer n; the term would then not strictly be polynomial, but we'd still say the function had "polynomial growth".

Case 2: Exponential → Linear

Suppose our original model was an exponential one:

$$y = ab^x$$

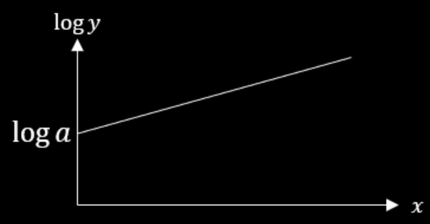
Then taking logs of both sides:

$$\log y = \log ab^x$$
$$\log y = \log a + x \log b$$

Again we can compare this against a straight line:

$$Y = mX + c$$

✏ If $y = ab^x$, then the graph of $\log y$ against x will be a straight line with gradient $\log b$ and vertical intercept $\log a$.



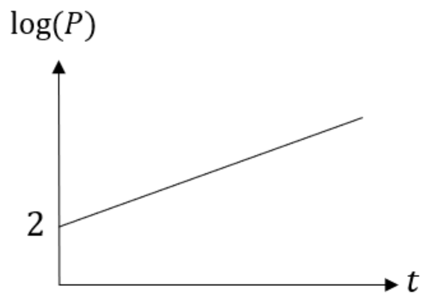
The key difference compared to Case 1 is that we're **only logging the y values** (e.g. number of transistors), not the x values (e.g. years elapsed). **Note that you <u>do not need to memorise</u> the contents of these boxes and we will work out from scratch each time…**

In summary, logging the y-axis **turns an exponential graph into a linear one**. Logging **both** the x and y-axis turns a polynomial graph into a linear one.

[Textbook] The graph represents the growth of a population of bacteria, P, over t hours. The graph has a gradient of 0.6 and meets the vertical axis at $(0,2)$ as shown.
A scientist suggests that this growth can be modelled by the equation $P = ab^t$, where a and b are constants to be found.

a. Write down an equation for the line.
b. Using your answer to part (a) or otherwise, find the values of a and b, giving them to 3 sf where necessary.

Interpret the meaning of the constant a in this model.



[Textbook] The table below gives the rank (by size) and population of the UK's largest cities and districts (London is number 1 but has been excluded as an outlier).

City	B'ham	Leeds	Glasgow	Sheffield	Bradford
Rank, R	2	3	4	5	6
Population, P	1 000 000	730 000	620 000	530 000	480 000

The relationship between the rank and population can be modelled by the formula:
$P = aR^n$ where a and n are constants.

Textbook Error: They use $R = aP^n$ but then plot $\log P$ against $\log R$.

a) Draw a table giving values of $\log R$ and $\log P$ to 2dp.
b) Plot a graph of $\log R$ against $\log P$ using the values from your table and draw the line of best fit.
c) Use your graph to estimate the values of a and n to two significant figures.

Dr Frost's wants to predict his number of Twitter followers P (@DrFrostMaths) t years from the start 2015. He predicts that his followers will increase exponentially according to the model $P = ab^t$, where a, b are constants that he wishes to find.
He records his followers at certain times. Here is the data:

Years t after 2015:	0.7	1.3	2.2
Followers P:	2353	3673	7162

a) Draw a table giving values of t and $\log P$ (to 3dp).
b) A line of best fit is drawn for the data in your new table, and it happens to go through the first data point above (where $t = 0.7$) and last (where $t = 2.2$).
Determine the equation of this line of best fit. (The y-intercept is 3.147)
c) Hence, determine the values of a and b in the model.
d) Estimate how many followers Dr Frost will have at the start of 2020 (when $t = 5$).

Exercise 14H Pg 331-333