U6 Chapter 9

Differentiation

Chapter Overview

1. Differentiate trigonometric, exponential and log functions.
2. Use chain, product and quotient rules.
3. Differentiate parametric equations.
4. Implicit Differentiation
5. Rates of change

Topics	What students need to learn:		Guidance
7 **Differentiation**	7.1	**Understand and use the derivative of $f(x)$ as the gradient of the tangent to the graph of $y = f(x)$ at a general point (x, y); the gradient of the tangent as a limit; interpretation as a rate of change**	**Know that $\frac{dy}{dx}$ is the rate of change of y with respect to x.** **The notation $f'(x)$ may be used for the first derivative and $f''(x)$ may be used for the second derivative.**
		sketching the gradient function for a given curve	**Given for example the graph of $y = f(x)$, sketch the graph of $y = f'(x)$ using given axes and scale. This could relate speed and acceleration for example.**
		second derivatives	
		differentiation from first principles for small positive integer powers of x and for $\sin x$ and $\cos x$	**For example, students should be able to use, for $n = 2$ and $n = 3$, the gradient expression** $$\lim_{h \to 0}\left(\frac{(x+h)^n - x^n}{h}\right)$$ **Students may use δx or h**

Topics	What students need to learn:		
	Content		Guidance
7 **Differentiation** *continued*	7.1 *cont.*	**Understand and use the second derivative as the rate of change of gradient;** connection to convex and concave sections of curves and points of inflection.	Use the condition $f''(x) > 0$ implies a minimum and $f''(x) < 0$ implies a maximum for points where $f'(x) = 0$ Know that at an inflection point $f''(x)$ changes sign. Consider cases where $f''(x) = 0$ and $f'(x) = 0$ where the point may be a minimum, a maximum or a point of inflection (e.g. $y = x^n$, $n > 2$)
	7.2	**Differentiate x^n, for rational values of n, and related constant multiples, sums and differences.**	**For example, the ability to differentiate expressions such as** $(2x + 5)(x - 1)$ **and** $\frac{x^2 + 3x - 5}{4x^{\frac{1}{2}}}$, $x > 0$, **is expected.**
		Differentiate e^{kx} and a^{kx}, $\sin kx$, $\cos kx$, $\tan kx$ and related sums, differences and constant multiples. Understand and use the derivative of $\ln x$	Knowledge and use of the result $\frac{d}{dx}(a^{kx}) = ka^{kx}\ln a$ is expected.
	7.3	**Apply differentiation to find gradients, tangents and normals**	**Use of differentiation to find equations of tangents and normals at specific points on a curve.**
		maxima and minima and stationary points. points of inflection	**To include applications to curve sketching. Maxima and minima problems may be set in the context of a practical problem.**
		Identify where functions are increasing or decreasing.	**To include applications to curve sketching.**

7 **Differentiation** *continued*	7.4	Differentiate using the product rule, the quotient rule and the chain rule, including problems involving connected rates of change and inverse functions.	Differentiation of cosec x, cot x and sec x. Differentiation of functions of the form $x = \sin y$, $x = 3\tan 2y$ and the use of $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)}$ Use of connected rates of change in models, e.g. $\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}t}$ Skill will be expected in the differentiation of functions generated from standard forms using products, quotients and composition, such as $2x^4 \sin x$, $\frac{e^{3x}}{x}$, $\cos^2 x$ and $\tan^2 2x$.
	7.5	Differentiate simple functions and relations defined implicitly or parametrically, for first derivative only.	The finding of equations of tangents and normals to curves given parametrically or implicitly is required.
	7.6	Construct simple differential equations in pure mathematics and in context, (contexts may include kinematics, population growth and modelling the relationship between price and demand).	Set up a differential equation using given information which may include direct proportion.

Differentiating trigonometric functions

You need to be able to differentiate sin x and cos x from first principles.

Example 1 **Prove, from first principles, that the derivative of sin x is cos x.**

$$\frac{d}{dx}(\sin x) = \mathbf{cos}\,\boldsymbol{x}$$

Things of helpfulness:

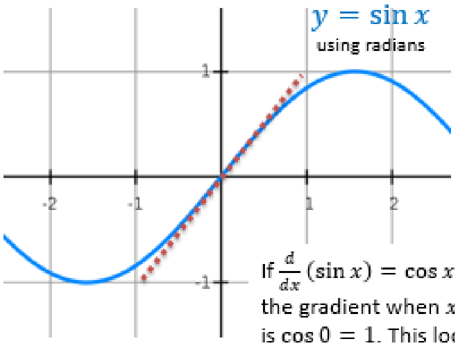
- As $x \to 0$, $\sin x \approx x$ and $\cos x \approx 1 - \frac{1}{2}x^2$
- $\sin(a+b) = \sin a \cos b + \cos a \sin b$

If $y = f(x)$ then $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

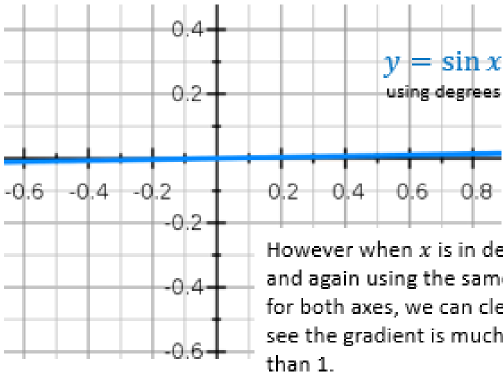
Why does this result only hold in radians?

Explanation 1:
The approximations $\sin x \approx x$ and $\cos x \approx 1 - \frac{1}{2}x^2$ (as $x \to 0$) only holds if x is in radians (we saw why in the chapter on radians). The proof that $\frac{d}{dx}(\sin x) = \cos x$ made use of these approximations.

Explanation 2:
We can see by observation if we look at the graph of $\sin x$ in radians and in degrees.



If $\frac{d}{dx}(\sin x) = \cos x$ then the gradient when $x = 0$ is $\cos 0 = 1$. This looks right from the graph.



However when x is in degrees, and again using the same scale for both axes, we can clearly see the gradient is much lower than 1.

$$\frac{d}{dx}(\sin kx) = k \cos kx$$

$$\frac{d}{dx}(\cos kx) = -k \sin kx$$

Quickfire Questions:

$\frac{d}{dx}(\sin 3x) =$

$\frac{d}{dx}(\cos 5x) =$

$\frac{d}{dx}(3 \sin 5x) =$

$\frac{d}{dx}(4 \cos 3x) =$

$\frac{d}{dx}\left(-\frac{1}{2}\sin x\right) =$

$\frac{d}{dx}\left(-\frac{2}{3}\cos\frac{1}{2}x\right) =$

Example

[Textbook] A curve has equation $y = \frac{1}{2}x - \cos 2x$. Find the stationary points on the curve in the interval $0 \leq x \leq \pi$.

Test Your Understanding

A curve has equation $y = \sin 3x + 2x$. Find the stationary points on the curve in the interval $0 \leq x \leq \frac{2}{3}\pi$.

Exercise 9A Page 234

Differentiation exponential and log functions

$$\frac{d}{dx}(e^x) = \boldsymbol{e^x}$$

$$\frac{d}{dx}(e^{kx}) = \boldsymbol{ke^{kx}}$$

$$\frac{d}{dx}(\ln x) = \boldsymbol{\frac{1}{x}}$$

$$\frac{d}{dx}(a^x) = \boldsymbol{a^x ln\ a}$$

$$\frac{d}{dx}(a^{kx}) = \boldsymbol{a^{kx} k\ ln\ a}$$

Quickfire Questions

$\frac{d}{dx}(3^x)$ =

$\frac{d}{dx}(x^3)$ =

$\frac{d}{dx}(\ln(3x))$ =

$\frac{d}{dx}(3^{2x}) =$

$\frac{d}{dx}(2^{3x}) =$

$\frac{d}{dx}(5\ln x) =$

$\frac{d}{dx}\left(e^{\frac{1}{2}x}\right) =$

$\frac{d}{dx}(5\ln(2x)) =$

$\frac{d}{dt}(9^t) =$

$\frac{d}{dx}\left(5(4^x)\right) =$

$\frac{d}{dx}(x^4) =$

$\frac{d}{dx}(\ln 6x) =$

$\frac{d}{dx}(6\ln x) =$

$\frac{d}{dx}(3e^{2x}) =$

$\frac{d}{dx}(e^{-x}) =$

'Meatier' Example:

A rabbit population P after t years can be modelled using $P = 1000(2^t)$. Determine after how many years the rate of population increase will reach 20,000 rabbits per year.

Test Your Understanding

1. **Differentiate $y = (e^x + 2)^2$** *(Hint: Expand first)*

2. A child has headlice and his parents treat it using a special shampoo. The population P of headlice after t days can be modelled using $P = 460(3^{-2t})$

a) Determine how many days have elapsed before the child has 20 headlice left.

b) Determine the rate of change of headlice after 3 days.

Exercise 9B Page 236-237

Differentiating combinations of functions

Functions can interact in different ways...

How to differentiate

1 Composite Function
i.e. of form $y = f(g(x))$

$$y = \sqrt{1+3x}$$

The 'outer' function here is the $\sqrt{}$ and the inner function the $1+3x$.
i.e. $f(x) = \sqrt{x}$ and $g(x) = 1+3x$

→ The Chain Rule (Ex9C)

2 Product of Two Functions
i.e. of form $y = f(x)g(x)$

$$y = x \sin 2x$$

→ The Product Rule (Ex9D)

3 Division (i.e. "Quotient") of Two Functions
i.e. of form $y = \frac{f(x)}{g(x)}$

$$y = \frac{\ln x}{x}$$

→ The Quotient Rule (Ex9E)

The Chain Rule

The Chain Rule:
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

The chain rule allows us to differentiate a composite function, i.e. a function within a function.

$$\text{Eg. } \boldsymbol{y = (3x^4 + x)^5}$$

Full Method:

Doing it mentally in one go:

(aka the 'bla method')

Further Practice

$$y = \left(x^2 + 1\right)^3$$

$$y = (\ln x)^3$$

$$y = e^{x^2+x}$$

$$y = (2^x + 1)^2$$

$$y = \ln(\sin x)$$

$$y = \sin 5x$$

$$y = \sin^2 x =$$

$$y = \sqrt{x+1} =$$

$$y = \cos^3 2x =$$

$$y = e^{e^x}$$

Test Your Understanding

C3 June 2011 Q1a

Differentiate with respect to x

(*a*) $\ln(x^2 + 3x + 5)$, **(2)**

[Textbook] Given that $y = \sqrt{5x^2 + 1}$, find $\frac{dy}{dx}$ at $(4,9)$

dx/dy

$$\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$$

Sometimes we might have x in terms of y, but we want to find $\frac{dy}{dx}$.

1. **Find $\frac{dy}{dx}$ when $x = 2y^2 + y$**

2. **Find the gradient of $x = (1 + 2y)^3$ when $y = 1$**

Exercise 9C Page 239-240

The Product Rule

As mentioned previously, the product rule is used, unsurprisingly, when we have a product of two functions.

The product rule:

If $y = uv$ then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

This is quite easy to remember. Differentiate one of the things but leave the other. Then do the other way round. Then add!
Since addition is commutative, it doesn't matter which way round we do it.

1. If $y = x^2 \sin x$, determine $\frac{dy}{dx}$

2. If $y = xe^{2x}$, determine the coordinates of the turning point.

Product + Chain Rule Examples

1. If $y = e^{4x}\sin^2 3x$,

show that $\frac{dy}{dx} = e^{4x}\sin 3x\,(A\cos 3x + B\sin 3x)$,

where A and B are constants to be determined.

2. Given that $f(x) = x^2\sqrt{3x-1}$, find $f'(x)$

Test Your Understanding

Edexcel C3 Jan 2012 Q1a

Differentiate with respect to x, giving your answer in its simplest form,

(a) $x^2 \ln(3x)$, **(4)**

Edexcel C3 June 2013 Q5(c)

$$\frac{dy}{dx} = \frac{1}{6x(x-1)^{\frac{1}{2}}}$$

Find $\frac{d^2y}{dx^2}$, simplifying your answer.

Exercise 9D Page 242

The Quotient Rule

Just as we use the 'product rule' to differentiate a 'product', we use the 'quotient rule' to differentiate a 'quotient' (i.e. division).

The quotient rule:

If $y = \frac{u}{v}$ then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Memorisation Tips: "Bottoms first!" The denominator (v) is the first term seen in the new denominator and numerator. The denominator gets squared. Note that in the numerator, we have $-$ instead of the $+$ seen in the Product Rule.

1. If $y = \frac{x}{2x+5}$, find $\frac{dy}{dx}$

2. Find the stationary point of $y = \frac{\sin x}{e^{2x}}$, $0 < x < \pi$

Test Your Understanding

Edexcel C3 Jan 2012 Q1a

Differentiate with respect to x, giving your answer in its simplest form,

(b) $\dfrac{\sin 4x}{x^3}$. **(5)**

Edexcel C3 June 2012 Q3

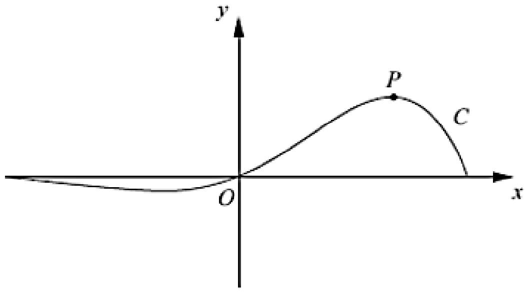


Figure 1

Figure 1 shows a sketch of the curve C which has equation

$$y = e^{x\sqrt{3}} \sin 3x, \quad -\frac{\pi}{3} \leq x \leq \frac{\pi}{3}.$$

(a) Find the x-coordinate of the turning point P on C, for which $x > 0$.
Give your answer as a multiple of π. **(6)**

(b) Find an equation of the normal to C at the point where $x = 0$. **(3)**

Exercise 9E Page 244-245

Differentiating other trigonometric functions

Differentiate $y = \tan x$

More generally: $\frac{d}{dx}(\tan kx) = k\sec^2 kx$

Differentiate $y = \sec x$

More generally: $\frac{d}{dx}(\sec kx) = k\sec kx\tan kx$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

$$\frac{d}{dx}(\operatorname{cosec} x) = -\,cosec\, x \cot x$$

Differentiate

(a) $y = \frac{cosec\ 2x}{x^2}$

(b) $y = \sec^3 x$

Test Your Understanding So Far

Edexcel C3 June 2013(R) Q5b

(*b*) Show that $\frac{\mathrm{d}}{\mathrm{d}x}(\sec^2 3x)$ can be written in the form

$$\mu(\tan 3x + \tan^3 3x)$$

where μ is a constant. **(3)**

Making use of $1 \div dx/dy$

Often in exam questions, you will be given x in terms of y, but want to find $\frac{dy}{dx}$ in terms of x.

The key is to make use of an appropriate trig identity, e.g:

$$\sin^2 x + \cos^2 x \equiv 1 \qquad 1 + \tan^2 x \equiv \sec^2 x$$

Eg. Given that $x = \tan y$, express $\frac{dy}{dx}$ in terms of x.

Exercise 9F Page 249-251

Further examples

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

1. **Show that if $y = \arcsin x$, then $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$**

2. **Given that $y = \arcsin x^2$ find $\frac{dy}{dx}$**

Test Your Understanding

Edexcel C3 Jan 2011 Q8b,c

Given that

$$x = \sec 2y,$$

(*b*) find $\dfrac{dx}{dy}$ in terms of y. **(2)**

(*c*) Hence find $\dfrac{dy}{dx}$ in terms of x. **(4)**

Eg. Given that $y = \arctan\left(\frac{1-x}{1+x}\right)$, find $\frac{dy}{dx}$

No Longer in Syllabus

Parametric Differentiation

Recall from the previous chapter that parametric equations are when we define each of x and y (and possibly z) in terms of some separate parameter, e.g. t.

If x and y are given as functions of a parameter t, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

1. Find the gradient at the point P where $t = 2$, on the curve given parametrically by

$$x = t^3 + t, \qquad y = t^2 + 1, \ \ t \in \mathbb{R}$$

2. Find the equation of the normal at the point P where $\theta = \frac{\pi}{6}$, to the curve with parametric equations
$x = 3\sin\theta\,, \ \ y = 5\cos\theta$

Test Your Understanding

C4 June 2012 Q6

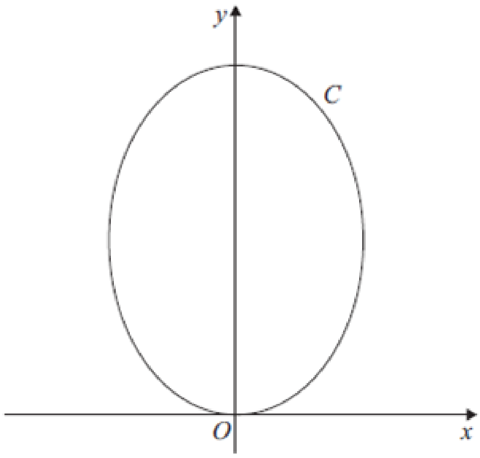


Figure 2

Figure 2 shows a sketch of the curve C with parametric equations

$$x = \sqrt{3}\sin 2t, \qquad y = 4\cos^2 t, \qquad 0 \leq t \leq \pi.$$

(*a*) Show that $\frac{dy}{dx} = k\sqrt{3}\tan 2t$, where k is a constant to be determined.

(5)

(*b*) Find an equation of the tangent to C at the point where $t = \frac{\pi}{3}$.

Give your answer in the form $y = ax + b$, where a and b are constants.

(4)

Exercise 9G Page 252-254

Implicit Differentiation

You're used to differentiating expressions where y is the subject, e.g. $y = x^2 + 3x$. The relationship between x and y is 'explicit' in the sense we can directly calculate y from x.

But what about **implicit** relations, e.g:

$$x^2 + y^2 = 8x \quad or \quad \cos(x + y) = \sin y$$

MIND-BLOWING MOMENT

$$y = x^2$$

$$\frac{d}{dx} \qquad \frac{dy}{dx} = 2x \qquad \frac{d}{dx}$$

When seeing $y = x^2$ and differentiating, you probably think you're just differentiating the x^2. But in fact, you're differentiating **both** sides of the equation! (with respect to x)

y (by definition) differentiates to $\frac{dy}{dx}$

To differentiate implicitly you only need to know 2 things:

- Differentiate each side of the equation (using chain rule if necessary).
- Remember that y differentiated with respect to x is, by definition, $\frac{dy}{dx}$

In general, when differentiating a function of y, but with respect to x, slap a $\frac{dy}{dx}$ on the end. i.e.

$$\frac{d}{dx}\left(f(y)\right) = f'(y)\frac{dy}{dx}$$

Examples

$$\frac{d}{dx}(y^2)$$

$$\frac{d}{dx}(\sin y)$$

$$\frac{d}{dx}(e^y)$$

$$\frac{d}{dx}(xy)$$

$$\frac{d}{dx}\left(e^{x^2y}\right)$$

$$\frac{d}{dx}(\tan(x+y))$$

$$\frac{d}{dx}(x^2+\cos y)$$

Meatier Examples

[Textbook] Find $\frac{dy}{dx}$ in terms of x and y where $x^3 + x + y^3 + 3y = 6$

[Textbook] Find the value of $\frac{dy}{dx}$ at the point $(1, 1)$, where $e^{2x} \ln y = x + y - 2$

Test Your Understanding

C4 Jan 2008 Q5

A curve is described by the equation

$$x^3 - 4y^2 = 12xy.$$

(*a*) Find the coordinates of the two points on the curve where $x = -8$. **(3)**

(*b*) Find the gradient of the curve at each of these points. **(6)**

C4 June 2014(R) Q3

$$x^2 + y^2 + 10x + 2y - 4xy = 10$$

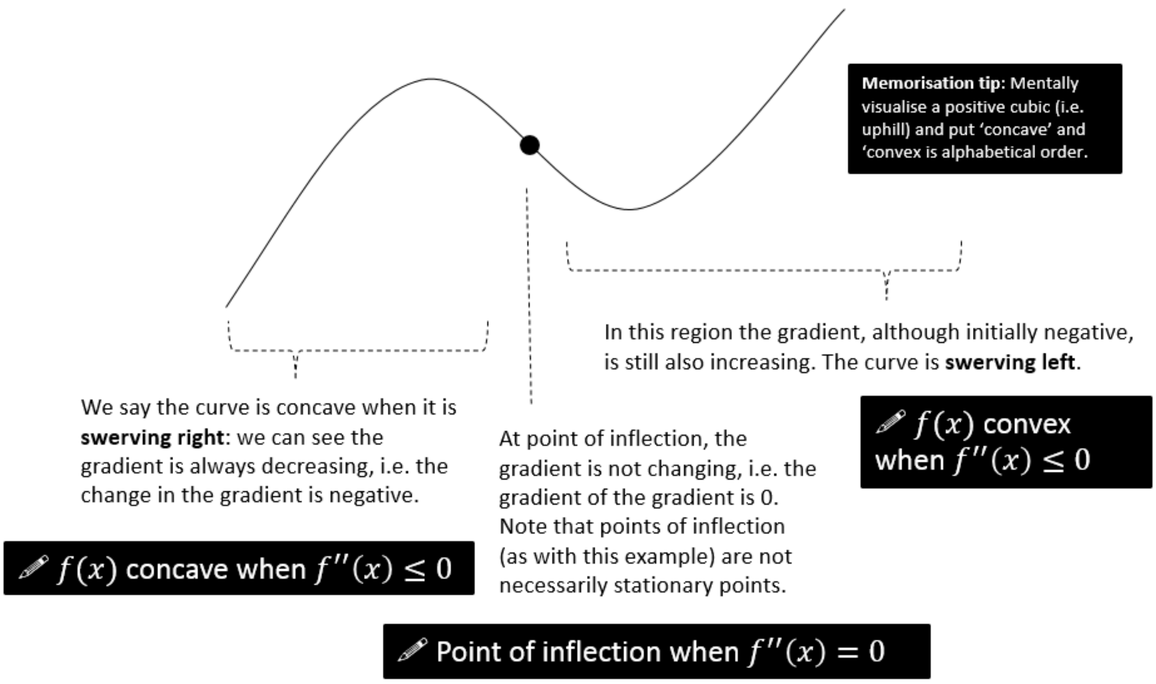
(*a*) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y, fully simplifying your answer. **(5)**

(*b*) Find the values of y for which $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. **(5)**

Exercise 9H Page 255-256

Using the second derivative

<u>Reminder</u>: a point of inflection is where the concavity of a curve changes, i.e. concave to convex or vice versa, or informally, 'swerving one way to swerving the other'.



Examples

1. **Find the interval on which the function $f(x) = x^3 + 4x + 3$ is concave.**

2. Show that $f(x) = e^{2x} + x^2$ is convex for all real values of x.

3. The curve C has equation $y = x^3 - 2x^2 - 4x + 5$. Find the coordinates of the point of inflection.

Exercise 9I Page 259-261

Relating Rates of Change

Eg. Determine the rate of change of the area A of a circle when the radius $r = 3$cm, given that the radius is changing at a rate of $5\ cm\ s^{-1}$

Firstly, how would we represent…

> **Fro Tip:** Whenever you see the word '<u>rate</u>', think $/dt$

"the rate of change of the area A"

"the rate of change of the radius r is 5"

"the area A of a circle"

Then by Chain Rule:

First copy top and bottom into the diagonals…

$$\frac{dA}{dt} = \frac{}{dr} \times \frac{dr}{}$$

…Then fill in the gaps with whatever variable you didn't use.

A differential equation is an equation that can be used to calculate a rate of change over time (essentially, what you have just been doing!)

Textbook. **In the decay of radioactive particles, the rate at which particles decay is proportional to the number of particles remaining. Write down a differential equation for the rate of change of the number of particles.**

Textbook. **Newton's law of cooling states that the rate of loss of temperature of a body is proportional to the excess temperature of the body compared to its surroundings. Write an equation that expresses this law.**

Textbook. **The head of a snowman of radius $R\ cm$ loses volume by evaporation at a rate proportional to its surface area. Assuming that the head is spherical, that the volume of a sphere is given by $V = \frac{4}{3}\pi R^3\ cm^3$ and that the surface area is $A = 4\pi R^2\ cm^2$, write down a differential equation for the rate of change of radius of the snowman's head.**

<u>Further Example</u>

Edexcel C4 June 2008 Q3

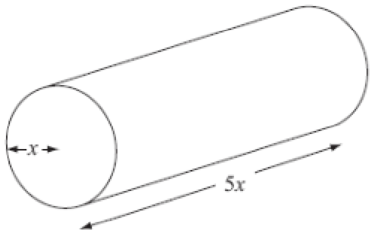


Figure 2 shows a right circular cylindrical metal rod which is expanding as it is heated. After t seconds the radius of the rod is x cm and the length of the rod is $5x$ cm.
The cross-sectional area of the rod is increasing at the constant rate of 0.032 cm^2 s^{-1}.

(*a*) Find $\frac{\mathrm{d}x}{\mathrm{d}t}$ when the radius of the rod is 2 cm, giving your answer to 3 significant figures. **(4)**

(*b*) Find the rate of increase of the volume of the rod when $x = 2$. **(4)**

Test Your Understanding

June 2012 Q2

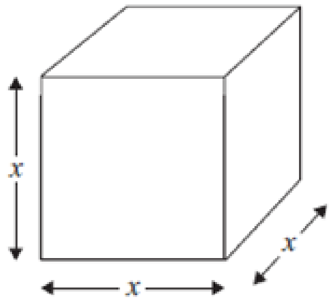


Figure 1

Figure 1 shows a metal cube which is expanding uniformly as it is heated.

At time t seconds, the length of each edge of the cube is x cm, and the volume of the cube is V cm^3.

(*a*) Show that $\dfrac{dV}{dx} = 3x^2$.

(1)

Given that the volume, V cm^3, increases at a constant rate of 0.048 cm^3 s^{-1},

(*b*) find $\dfrac{dx}{dt}$ when $x = 8$,

(2)

(*c*) find the rate of increase of the total surface area of the cube, in cm^2 s^{-1}, when $x = 8$.

(3)

Exercise 9J Page 263-264