P2 Chapter 8 Parametric Equations

Chapter Overview

This chapter is very similar to the trigonometry chapters in Year 1. The only difference is that new trig functions: sec, $cosec$ and cot, are introduced.

1:: Converting from parametric to Cartesian form.

If $x = 2\cos t + 1$ and $y = 3\sin t$, find a Cartesian equations connecting x and y.

2:: Sketching parametric curves.

Sketch the curve with parametric equations $x = 2t$ and $y = \frac{5}{t}$.

3:: Finding points of intersection.

Curve C_1 has the parametric equations $x = t^2$ and $y = 4t$. The curve C_2 has the Cartesian equation $x + y + 4 = 0$. The two curves intersect at A. Find the coordinates of A.

4:: Modelling

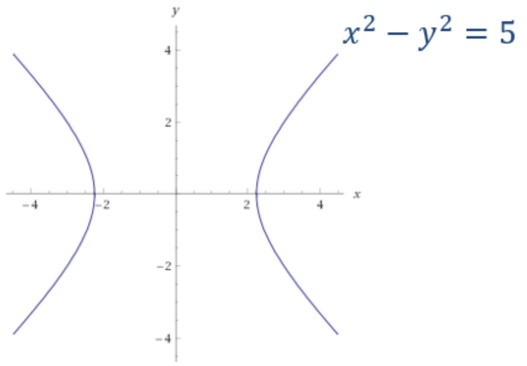
A plane's position at time t seconds after take-off can be modelled with the parametric equations:

$x = (v\cos\theta)t$ m, $\quad y = (v\sin\theta)t$ m, $\quad t > 0$

…

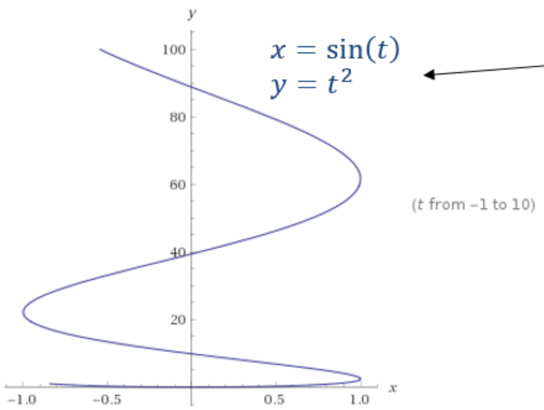
Topics	What students need to learn:		
	Content		Guidance
3 **Coordinate geometry in the (x, y) plane** *continued*	3.3	Understand and use the parametric equations of curves and conversion between Cartesian and parametric forms.	For example: $x = 3\cos t,\ y = 3\sin t$ describes a circle centre O radius 3 $x = 2 + 5\cos t,\ y = -4 + 5\sin t$ describes a circle centre $(2, -4)$ with radius 5 $x = 5t,\ y = \frac{5}{t}$ 5describes the curve $xy = 25$ (or $y = \frac{25}{x}$) $x = 5t, y = 3t^2$ describes the quadratic curve $25y = 3x^2$ and other familiar curves covered in the specification. Students should pay particular attention to the domain of the parameter t, as a specific section of a curve may be described.
	3.4	Use parametric equations in modelling in a variety of contexts.	A shape may be modelled using parametric equations or students may be asked to find parametric equations for a motion. For example, an object moves with constant velocity from $(1, 8)$ at $t = 0$ to $(6, 20)$ at $t = 5$. This may also be tested in Paper 3, section 7 (kinematics).

What are they and what is the point?



Typically, with two variables x and y, we can relate the two by **a single equation involving just x and y**.
This is known as a **Cartesian equation**.

The line shows all points (x, y) which satisfy the Cartesian equation.



However, in Mechanics for example, we might want each of the x and y values to be some function of time t, as per this example.

This would allow us to express the position of a particle at time t as the vector:

$$\begin{pmatrix} \sin t \\ t^2 \end{pmatrix}$$

These are known as **parametric equations**, because **each of x and y are defined in terms of some other variable, known as the parameter** (in this case t).

Converting parametric to Cartesian

How could we convert these parametric equations into a single Cartesian one?

$$x = 2t, \qquad y = t^2, \qquad -3 < t < 3$$

What is the domain of the function?

✏ **If $x = p(t)$ and $y = q(t)$ can be written as $y = f(x)$, then the domain of f is the range of p...**

✏ **and the range of f is the range of q.**

Further Example

[Textbook] A curve has the parameter equations

$$x = \ln(t+3)\,, \qquad y = \frac{1}{t+5}, \qquad t > -2$$

a) Find a Cartesian equation of the curve of the form $y = f(x)$, $x > k$, where k is a constant to be found.
b) Write down the range of $f(x)$.

A common strategy for domain/range questions is to consider what happens are the boundary value (in this case -2), then since $t > -2$, consider what happens as t increases.

Test Your Understanding

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The curve C has parametric equations

$$x = \ln(t+2), \quad y = \frac{1}{(t+1)}, \quad t > -1.$$

(c) Find a cartesian equation of the curve C, in the form $y = \mathrm{f}(x)$. **(4)**

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6. The curve C has parametric equations

$$x = \ln t, \qquad y = t^2 - 2, \qquad t > 0.$$

(b) a cartesian equation of C.

(3)

…when you have trig identities

Exercise 8A Page 200

When we have trig functions we have to use identities to find the Cartesian equation. Generally we use $\sin^2 t + \cos^2 t \equiv 1$ or $1 + \tan^2 t \equiv \sec^2 t$

[Textbook] A curve has the parametric sequences $x = \sin t + 2$, $y = \cos t - 3$, $t \in \mathbb{R}$.

a) Find a Cartesian equation for the curve.

b) Hence sketch the curve.

[Textbook] A curve is defined by the parametric equations

$$x = \sin t\,, \qquad y = \sin 2t\,, \qquad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$$

a) Find a Cartesian equation of the curve in the form $y = f(x)$, $-k \leq x \leq k$, stating the value of the constant k.

b) Write down the range of $f(x)$.

Test Your Understanding

C4 June 2013

which double angle formula would be best here?

4. A curve C has parametric equations

$$x = 2\sin t, \qquad y = 1 - \cos 2t, \qquad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$$

(*b*) Find a cartesian equation for C in the form

$$y = \text{f}(x), \qquad -k \le x \le k,$$

stating the value of the constant k.

[Textbook] A curve C has parametric equations

$$x = \cot t + 2, \qquad y = cosec^2\, t - 2, \qquad 0 < t < \pi$$

a) Find the equation of the curve in the form $y = f(x)$ and state the domain of x for which the curve is defined.

b) Hence, sketch the curve.

Exercise 8B Page 204

Sketching Parametric Curves

Input interpretation:

plot	$x = \theta\cos(\theta)$ $y = \theta\sin(\theta)$	$\theta = 0$ to 4π

$\boldsymbol{\theta}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3}{4}\pi$	π	$\frac{5}{4}\pi$	$\frac{3}{2}\pi$	$\frac{7}{4}\pi$	2π
$\boldsymbol{x}$									
$\boldsymbol{y}$									

Test Your Understanding

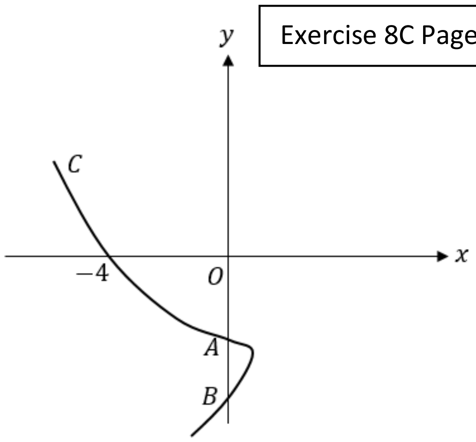
[Textbook] Draw the curve given by the parametric equations $x = 2t$, $y = t^2$, for $-1 \leq t \leq 5$.

Points of Intersection

We can find where a parametric curve crosses a particular axis or where curves cross each other.
The key is to first find the value of the parameter t.

Exercise 8C Page 207

[Textbook] The diagram shows a curve C with parametric equations $x = at^2 + t, \quad y = a(t^3 + 8)$, $t \in \mathbb{R}$, where a is a non-zero constant. Given that C passes through the point $(-4,0)$,
a) find the value of a.
b) find the coordinates of the points A and B where the curve crosses the y-axis.



[Textbook] A curve is given parametrically by the equations $x = t^2,\ y = 4t$. The line $x + y + 4 = 0$ meets the curve at A. Find the coordinates of A.

Whenever you want to solve a Cartesian equation and pair of parametric equations simultaneously, substitute the parametric equations into the Cartesian one.

[Textbook] The diagram shows a curve C with parametric equations

$$x = \cos t + \sin t\,, \qquad y = \left(t - \frac{\pi}{6}\right)^2, \qquad -\frac{\pi}{2} < t < \frac{4\pi}{3}$$

a) Find the point where the curve intersects the line $y = \pi^2$.
b) Find the coordinates of the points A and B where the curve cuts the y-axis.

Test Your Understanding

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5.

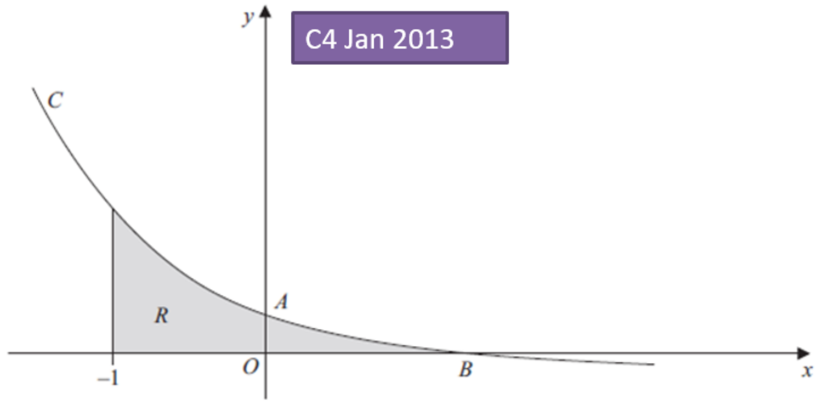


Figure 2

Figure 2 shows a sketch of part of the curve C with parametric equations

$$x = 1 - \frac{1}{2}t, \qquad y = 2^t - 1.$$

The curve crosses the y-axis at the point A and crosses the x-axis at the point B.

(*a*) Show that A has coordinates (0, 3).

(2)

(*b*) Find the x-coordinate of the point B.

(2)

Exercise 8D Page 211

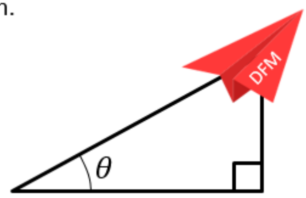
Modelling

As we saw at the start of this chapter, parametric equations are frequently used in mechanics, particularly where the (x, y) position (the Cartesian variables) depends on time t (the parameter).

[Textbook] A plane's position at time t seconds after take-off can be modelled with the following parametric equations:

$$x = (v\cos\theta)t \text{ m}, \qquad y = (v\sin\theta)t \text{ m}, \qquad t > 0$$

where v is the speed of the plane, θ is the angle of elevation of its path, x is the horizontal distance travelled and y is the vertical distance travelled, relative to a fixed origin.
When the plane has travelled 600m horizontally, it has climbed 120m.
a. find the angle of elevation, θ.
Given that the plane's speed is 50 m s^{-1},
b. find the parametric equations for the plane's motion.
c. find the vertical height of the plane after 10 seconds.
d. show that the plane's motion is a straight line.
e. explain why the domain of t, $t > 0$, is not realistic.



Further Example

[Textbook] The motion of a figure skater relative to a fixed origin, O, at time t minutes is modelled using the parametric equations

$$x = 8\cos 20t\,, \qquad y = 12\sin\left(10t - \frac{\pi}{3}\right), \qquad t \geq 0$$

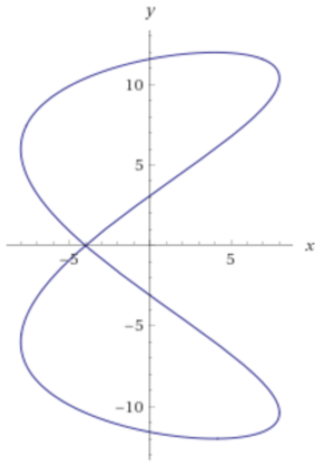
where x and y are measured in metres.

a) Find the coordinates of the figure skater at the beginning of his motion.

b) Find the coordinates of the point where the figure skater intersects his own path.

c) Find the coordinates of the points where the path of the figure skater crosses the y-axis.

d) Determine how long it takes the figure skater to complete one complete figure-of-eight motion.



Exercise 8E Page 218