Pure 2

Vectors

<u>Chapter Overview</u>

1:: Distance between two points.

2:: i, j, k notation for vectors

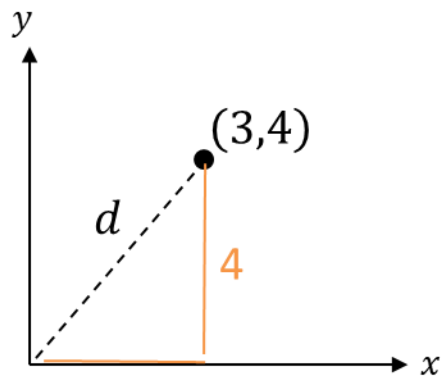
3:: Magnitude of a 3D vector and using it to find angle between vector and a coordinate axis.

4:: Solving Geometric Problems

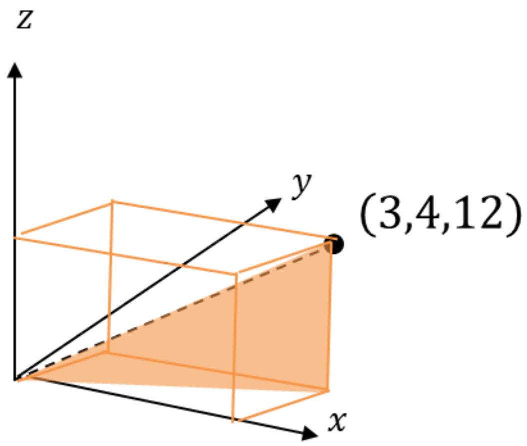
5:: Application to Mechanics

Topics	What students need to learn:		
	Content		Guidance
10 **Vectors**	10.1	**Use vectors in two dimensions** and in three dimensions	**Students should be familiar with column vectors and with the use of i and j unit vectors in two dimensions** and **i**, **j** and **k** unit vectors in three dimensions.
	10.2	**Calculate the magnitude and direction of a vector and convert between component form and magnitude/direction form.**	**Students should be able to find a unit vector in the direction of a, and be familiar with the notation $\|a\|$.**
	10.3	**Add vectors diagrammatically and perform the algebraic operations of vector addition and multiplication by scalars, and understand their geometrical interpretations.**	**The triangle and parallelogram laws of addition.** **Parallel vectors.**
	10.4	**Understand and use position vectors; calculate the distance between two points represented by position vectors.**	$\overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ **The distance d between two points (x_1, y_1) and (x_2, y_2) is given by** $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$
	10.5	**Use vectors to solve problems in pure mathematics and in context, (including forces).**	**For example, finding position vector of the fourth corner of a shape (e.g. parallelogram) $ABCD$ with three given position vectors for the corners A, B and C.** **Or use of ratio theorem to find position vector of a point C dividing AB in a given ratio.** **Contexts such as velocity, displacement, kinematics and forces will be covered in Paper 3, Sections 6.1, 7.3 and 8.1 – 8.4**

Distance from the origin and magnitude of a vector



In 2D, how did we find the distance from a point to the origin?



How about in 3D then?

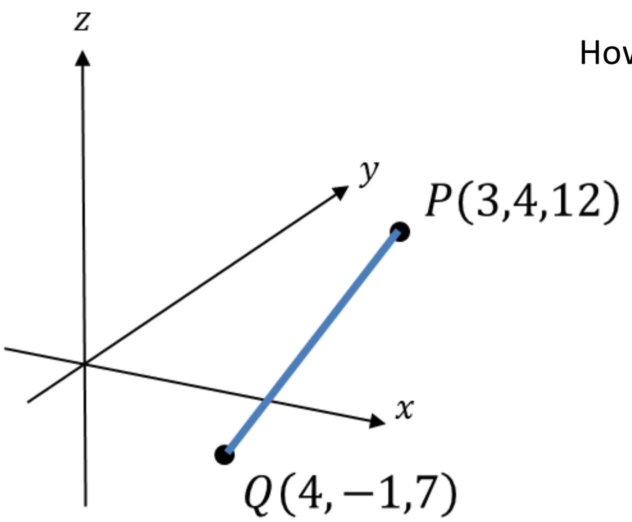
The magnitude of a vector $\boldsymbol{a} = \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \end{pmatrix}$**:**

$$|\boldsymbol{a}| = \sqrt{\boldsymbol{x}^2 + \boldsymbol{y}^2 + \boldsymbol{z}^2}$$

And the distance of $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ **from the origin is**

$$\sqrt{\boldsymbol{x}^2 + \boldsymbol{y}^2 + \boldsymbol{z}^2}$$

Distance between two 3D points



How do we find the distance between P and Q?

The distance between two points is:

$$\boldsymbol{d} = \sqrt{(\boldsymbol{\Delta x})^2 + (\boldsymbol{\Delta y})^2 + (\boldsymbol{\Delta z})^2}$$

Δx means "change in x"

Quickfire Questions:

Distance of $(4,0,-2)$ from the origin:

$$\left|\begin{pmatrix} 5 \\ 4 \\ -1 \end{pmatrix}\right| =$$

Distance between $(0,4,3)$ and $(5,2,3)$.

Distance between $(1,1,1)$ and $(2,1,0)$.

Distance between $(-5,2,0)$ and $(-2,-3,-3)$.

Tip: Because we're squaring, it doesn't matter whether the change is negative or positive.

Test Your Understanding So Far…

[Textbook] **Find the distance from the origin to the point $P(7,7,7)$.**

[Textbook] **The coordinates of A and B are $(5,3,-8)$ and $(1,k,-3)$ respectively. Given that the distance from A to B is $3\sqrt{10}$ units, find the possible values of k.**

Ex 12A p.338

$\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ notation

In 2D you were previously introduced to $\boldsymbol{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\boldsymbol{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as unit vectors in each of the x and y directions.

It meant for example that $\begin{pmatrix} 8 \\ -2 \end{pmatrix}$ could be written as $8\boldsymbol{i} - 2\boldsymbol{j}$ since $8\begin{pmatrix} 1 \\ 0 \end{pmatrix} - 2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ -2 \end{pmatrix}$

Unsurprisingly, **in 3D:**

$$\boldsymbol{i} = \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \ \boldsymbol{j} = \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \end{pmatrix}, \ \boldsymbol{k} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{pmatrix}$$

Quickfire Questions

1. Put in i, j, k notation:

$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} =$

$\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} =$

2. Write as a column vector:

$4\boldsymbol{j} + \boldsymbol{k} =$

$\boldsymbol{i} - \boldsymbol{j} =$

3. If $A(1,2,3),\ B(4,0,-1)$ then

$\overrightarrow{AB} =$

4. If $\boldsymbol{a} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}$ then $3\boldsymbol{a} + 2\boldsymbol{b} =$

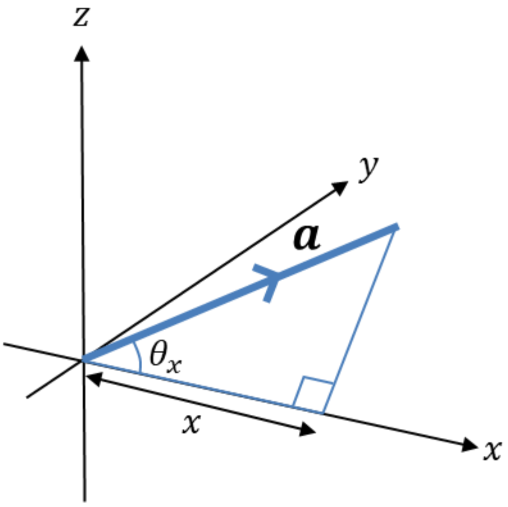
Examples

1. Find the magnitude of $\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k}$ and hence find $\widehat{\boldsymbol{a}}$, the unit vector in the direction of $\boldsymbol{a}$.

2. If $\boldsymbol{a} = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix}$ is $2\boldsymbol{a} - 3\boldsymbol{b}$ parallel to $4\boldsymbol{i} - 5\boldsymbol{k}$?

Angles between vectors and an axis

How could you work out the angle between a vector and the x-axis?



The angle between $\boldsymbol{a} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and the x-axis is:

$$\cos \theta_x = \frac{x}{|\boldsymbol{a}|}$$

and similarly for the y and z axes.

[Textbook] **Find the angles that the vector $\boldsymbol{a} = 2\boldsymbol{i} - 3\boldsymbol{j} - \boldsymbol{k}$ makes with each of the positive coordinate axis.**

Test Your Understanding

[Textbook] **The points A and B have position vectors $4\boldsymbol{i} + 2\boldsymbol{j} + 7\boldsymbol{k}$ and $3\boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k}$ relative to a fixed origin, $\boldsymbol{O}$. Find $\overrightarrow{AB}$ and show that ΔOAB is isosceles.**

(a) Find the angle that the vector $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ makes with the x-axis.

(b) By similarly considering the angle that $\boldsymbol{b} = \boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}$ makes with the x-axis, determine the area of OAB where $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$. (Hint: draw a diagram)

Ex 12B p.341-343

<u>Solving geometric problems</u>

For more general problems involving vectors, often **<u>drawing a diagram</u>** helps!

[Textbook] **$\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{D}$ are the points $(\mathbf{2}, -\mathbf{5}, -\mathbf{8})$, $(\mathbf{1}, -\mathbf{7}, -\mathbf{3})$, $(\mathbf{0}, \mathbf{15}, -\mathbf{10})$ and $(\mathbf{2}, \mathbf{19}, -\mathbf{20})$ respectively.**

a. **Find $\overrightarrow{\boldsymbol{AB}}$ and $\overrightarrow{\boldsymbol{DC}}$, giving your answers in the form $\boldsymbol{pi} + \boldsymbol{qj} + \boldsymbol{rk}$.**

b. **Show that the lines $\boldsymbol{AB}$ and $\boldsymbol{DC}$ are parallel and that $\overrightarrow{\boldsymbol{DC}} = \mathbf{2}\overrightarrow{\boldsymbol{AB}}$.**

c. **Hence describe the quadrilateral $\boldsymbol{ABCD}$.**

[Textbook] **$\boldsymbol{P}$, $\boldsymbol{Q}$ and $\boldsymbol{R}$ are the points $(\mathbf{4}, -\mathbf{9}, -\mathbf{3})$, $(\mathbf{7}, -\mathbf{7}, -\mathbf{7})$ and $(\mathbf{8}, -\mathbf{2}, \mathbf{0})$ respectively. Find the coordinates of the point $\boldsymbol{S}$ so that $\boldsymbol{PQRS}$ forms a parallelogram.**

There are many contexts in maths where we can 'compare coefficients', e.g.

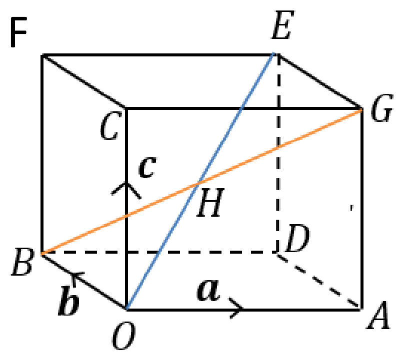
$$3x^2 + 5x \equiv A(x^2 + 1) + Bx + C$$

Comparing x^2 terms: $3 = A$

We can do the same with vectors:

[Textbook] **Given that**
$\mathbf{3}\boldsymbol{i} + (\boldsymbol{p} + \mathbf{2})\boldsymbol{j} + \mathbf{120}\boldsymbol{k} = \boldsymbol{pi} - \boldsymbol{qj} + \mathbf{4}\boldsymbol{pqrk}$, find the values of $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{r}$.

[Textbook] **The diagram shows a cuboid whose vertices are O, A, B, C, D, E, F and G. Vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are the position vectors of the vertices A, B and C respectively. Prove that the diagonals OE and BG bisect each other.**



The strategy behind this type of question is to find the point of intersection in 2 ways, and compare coefficients.

Application to Mechanics

Out of displacement, speed, acceleration, force, mass and time, all but mass and time are vectors. Clearly these can act in 3D space.

	Vector	Scalar
Force	$\begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} N$	
Acceleration	$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} ms^{-2}$	
Displacement	$\begin{pmatrix} 12 \\ 3 \\ 4 \end{pmatrix} m$	
Velocity	$\begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix} ms^{-1}$	

Example

[Textbook] **A particle of mass 0.5 kg is acted on by three forces.**

$$\boldsymbol{F_1} = (2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k})\ \boldsymbol{N} \boldsymbol{F_2} = (-\boldsymbol{i} + 3\boldsymbol{j} - 3\boldsymbol{k})\ \boldsymbol{N} \boldsymbol{F_3} = (4\boldsymbol{i} - 3\boldsymbol{j} - 2\boldsymbol{k})\ \boldsymbol{N}$$

a. **Find the resultant force $\boldsymbol{R}$ acting on the particle.**

b. **Find the acceleration of the particle, giving your answer in the form $(\boldsymbol{pi} + \boldsymbol{qj} + \boldsymbol{rk})$ ms^{-2}.**

c. **Find the magnitude of the acceleration.**

Given that the particle starts at rest,

d. Find the distance travelled by the particle in the first 6 seconds of its motion.

Ex 12D p.348-349