Upper 6 Chapter 11

Integration

Chapter Overview

1. Integration
2. Integrals of the form $f'(ax+b)$
3. Using Trigonometric Identities
4. The Reverse Chain Rule
5. Integration by Substitution
6. Integration by Parts
7. Integration Using Partial Fractions
8. Area Under a Curve
9. The Trapezium Rule
10. Parametric Equations
11. Differential Equations
12. Forming Differential Equations

8.1	**Know and use the Fundamental Theorem of Calculus**	**Integration as the reverse process of differentiation. Students should know that for indefinite integrals a constant of integration is required.**
8.2	**Integrate x^n (excluding $n = -1$) and related sums, differences and constant multiples.** Integrate e^{kx}, $\frac{1}{x}$, $\sin kx$, $\cos kx$ and related sums, differences and constant multiples.	**For example, the ability to integrate expressions such as $\frac{1}{2}x^2 - 3x^{-\frac{1}{2}}$ and $\frac{(x+2)^2}{x^{\frac{1}{2}}}$ is expected.** x **Given $f'(x)$ and a point on the curve, Students should be able to find an equation of the curve in the form $y = f(x)$.** To include integration of standard functions such as $\sin 3x$, $\sec^2 2x$, $\tan x$, e^{5x}, $\frac{1}{2x}$. Students are expected to be able to use trigonometric identities to integrate, for example, $\sin^2 x$, $\tan^2 x$, $\cos^2 3x$.

8.3	**Evaluate definite integrals; use a definite integral to find the area under a curve** and the area between two curves	Students will be expected to be able to evaluate the area of a region bounded by a curve and given straight lines, or between two curves. This includes curves defined parametrically. For example, find the finite area bounded by the curve $y = 6x - x^2$ and the line $y = 2x$ Or find the finite area bounded by the curve $y = x^2 - 5x + 6$ and the curve $y = 4 - x^2$.
8.4	Understand and use integration as the limit of a sum.	Recognise $\int_a^b f(x)\,dx = \lim_{\delta x \to 0} \sum_{x=a}^{b} f(x)\,\delta x$
8.5	Carry out simple cases of integration by substitution and integration by parts; understand these methods as the inverse processes of the chain and product rules respectively (Integration by substitution includes finding a suitable substitution and is limited to cases where one substitution will lead to a function which can be integrated; integration by parts includes more than one application of the method but excludes reduction formulae.)	Students should recognise integrals of the form $\int \frac{f'(x)}{f(x)}\,dx = \ln f(x) + c$. The integral $\int \ln x\,dx$ is required Integration by substitution includes finding a suitable substitution and is limited to cases where one substitution will lead to a function which can be integrated; integration by parts includes more than one application of the method but excludes reduction formulae.

8.3	**Evaluate definite integrals; use a definite integral to find the area under a curve** and the area between two curves	Students will be expected to be able to evaluate the area of a region bounded by a curve and given straight lines, or between two curves. This includes curves defined parametrically. For example, find the finite area bounded

INTEGRATION

Integration is the reverse of differentiation. We use known derivatives to integrate.

The following are integrals that you should know:

SKILL #1: Integrating Standard Functions

There's certain results you should be able to integrate straight off, by just thinking about the opposite of differentiation.

y	$\int y\,dx$
x^n	
e^x	
$\frac{1}{x}$	
$\cos x$	
$\sin x$	
$\sec^2 x$	
$cosec\ x \cot x$	
$cosec^2 x$	
$\sec x \tan x$	

The $|x|$ has to do with problems when x is negative (when $\ln x$ is not defined)

Remember my memorisation trick of picturing sin above cos from C3, so that 'going down' is differentiating and 'going up' is integrating, and we change the sign if the wrong way round.

It's vital you remember this one.

Have a good stare at this slide before turning your paper over – let's see how many you remember…

Quickfire Questions (without cheating!)

$$\int \sec x \tan x \; dx = \boxed{}$$

$$\int \sin x \; dx = \boxed{}$$

$$\int cosec^2\, x \, dx = \boxed{}$$

$$\int -\cos x \; dx = \boxed{}$$

Quickfire Questions (without cheating!)

$$\int \sec^2 x \ dx = \boxed{}$$

$$\int cosec\ x \cot x \ dx = \boxed{}$$

$$\int \frac{1}{x} \ dx = \boxed{}$$

$$\int -\sin x \ dx = \boxed{}$$

Quickfire Questions (without cheating!)

$$\int cosec^2\, x\, dx = \boxed{}$$

$$\int \sin x \ dx = \boxed{}$$

$$\int \sec x \tan x \ dx = \boxed{}$$

$$\int \cos x \ dx = \boxed{}$$

Test Your Understanding

$$\int 2\cos x + \frac{3}{x} - \sqrt{x}\,dx =$$

$$\int \frac{\cos x}{\sin^2 x}\,dx = \int$$

Hint: What 'reciprocal' trig functions does this simplify to?

[Textbook] Given that $\int_a^{3a} \frac{2x+1}{x}\,dx = \ln 12$, find the exact value of a.

Important Notes:
We can simplify:
$$\frac{x+1}{x} \equiv \frac{x}{x} + \frac{1}{x} \equiv 1 + \frac{1}{x}$$
However it is **NOT** true that:
$$\frac{x}{x+1} \equiv \frac{x}{x} + \frac{x}{1}$$
In my experience students often fail to spot when they can split up a fraction to then integrate.

Exercise 11A Page 295

INTEGRALS OF THE FORM $f'(ax+b)$

The following are integrals that you should know:

$\int x^n \, dx$	=	$\frac{x^{n+1}}{n+1} + c$
$\int e^x \, dx$	=	$e^x + c$
$\int \frac{1}{x} dx$	=	$ln\lvert x\rvert + c$
$\int \cos x \, dx$	=	$\sin x + c$
$\int \sin x \, dx$	=	$-\cos x + c$
$\int \sec^2 x \, dx$	=	$\tan x + c$
$\int cosec\, x \cot x \, dx$	=	$-cosec\, x + c$
$\int cosec^2 x \, dx$	=	$-\cot x + c$
$\int \sec x \tan x \, dx$	=	$\sec x + c$

SKILL #2: Integrating $f(ax+b)$

$$\frac{d}{dx}(\sin(3x+1)) = \boxed{}$$

Therefore:

$$\int \cos(3x+1) \ dx = \boxed{}$$

✎ For any expression where inner function is $ax+b$, integrate as before and $\div\, a$.

$$\int f'(ax+b)dx = \frac{1}{a}f(ax+b) + C$$

Quickfire:

$$\int e^{3x} \, dx = \boxed{}$$

$$\int \frac{1}{5x+2} dx = \boxed{}$$

$$\int 2\sec^2(3x-2) \, dx = \boxed{}$$

Fro Tip: For $\int (ax+b)^n \, dx$, ensure you divide by the $(n+1)$ and the a

$$\int (3x+4)^3 \, dx = \boxed{}$$

$$\int \sin(1-5x) \, dx = \boxed{}$$

$$\int \frac{1}{3(4x-2)^2} dx = \boxed{}$$

$$\int (10x+11)^{12} = \boxed{}$$

Check Your Understanding

$$\int e^{3x+1}dx =$$

$$\int \frac{1}{1-2x}dx =$$

$$\int (4-3x)^5 dx =$$

$$\int \sec(3x)\tan(3x)\, dx =$$

Exercise 11B

Pages 297-298

1 a $\int \sin(2x+1)\,dx =$

c $\int 4e^{x+5}\,dx =$

e $\int cosec^2 3x\,dx =$

f $\int \sec 4x \tan 4x \; dx =$

g $\int 3\sin\left(\frac{1}{2}x+1\right)dx =$

h $\int cosec\, 2x \cot 2x \; dx =$

2 a $\int e^{2x} - \frac{1}{2}\sin(2x-1)\,dx$

$=$

b $\int (e^x+1)^2\,dx =$

c $\int \sec^2 2x\,(1+\sin 2x)\,dx =$

d $\int \frac{3-2\cos\left(\frac{1}{2}x\right)}{\sin^2\left(\frac{1}{2}x\right)}\,dx =$

e $\int e^{3-x} + \sin(3-x) + \cos(3-x)\,dx$

$=$

3 a $\int \frac{1}{2x+1}dx =$

b $\int \frac{1}{(2x+1)^2}dx =$

c $\int (2x+1)^2\,dx =$

d $\int \frac{3}{4x-1}dx =$

f $\int \frac{3}{(1-4x)^2}dx =$

h $\int \frac{3}{(1-2x)^3}dx =$

j $\int \frac{5}{3-2x}dx =$

4 a $\int 3\sin(2x+1) + \frac{4}{2x+1}\;dx$

$=$

c $\int \frac{1}{\sin^2 2x} + \frac{1}{1+2x} + \frac{1}{(1+2x)^2}dx$

$=$

USING TRIGONOMETRIC IDENTITIES

The following are identities that you should know:

$\boldsymbol{sin(A \pm B)}$	=	
$\boldsymbol{cos(A \pm B)}$	=	
$\boldsymbol{tan(A \pm B)}$	=	
$\boldsymbol{sin\ 2A}$	=	
$\boldsymbol{cos\ 2A}$	=	
$\boldsymbol{cos\ 2A}$	=	
$\boldsymbol{cos\ 2A}$	=	
$\boldsymbol{tan\ 2A}$	=	
$\boldsymbol{sec^2 A}$	=	
$\boldsymbol{cosec^2 A}$	=	

We can use these identities to transform an expression that *cannot* be integrated into one that *can* be integrated.

These first examples focus on manipulation of the identities rather than integration.

Examples

1) $sin\ 4x =$

2) $2\ sin\ 3x \cos 3x =$

3) $cos\ 5x =$

4) $4\ cos^2 3x - 2 =$

SKILL #3: Integrating using Trig Identities

Some expressions, such as $\sin^2 x$ and $\sin x \cos x$ can't be integrated directly, but we can use one of our trig identities to replace it with an expression we can easily integrate.

Q Find $\int \sin^2 x \ dx$

Q Find $\int \sin 3x \cos 3x \ dx$

Q Find $\int \cos^2 x \ dx$

Q Find $\int \tan^2 x \ dx$

Check Your Understanding

Q Find $\int (\sec x + \tan x)^2 \ dx$

Q Find $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^2 3x \ dx$

Further examples

Show that

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{8}} \sin^2 x \, dx = \frac{\pi}{48} + \frac{1-\sqrt{2}}{8}$$

Exercise 11C Page 300

SKILL #4: Reverse Chain Rule

There's certain more complicated expressions which look like the result of having applied the chain rule. I call this process '**consider then scale**':

1. **Consider some expression that will differentiate to something similar to it.**
2. **Differentiate, and adjust for any scale difference.**

$\int x(x^2+5)^3 \, dx$	$\int \cos x \sin^2 x \, dx$	$\int \frac{2x}{x^2+1} \, dx$
The first x looks like it arose from differentiating the x^2	The $\cos x$ probably arose from differentiating the sin.	The $2x$ probably arose from differentiating the x^2.

Integration by Inspection/Reverse Chain Rule: Use common sense to **consider some expression** that would differentiate to the expression given. Then **scale** appropriately.
Common patterns:

$$\int k\frac{f'(x)}{f(x)}dx \rightarrow \quad Try\ \ln|f(x)|$$

In words: "If the bottom of a fraction differentiates to give the top (forgetting scaling), try ln of the bottom".

$$\int kf'(x)[f(x)]^n \rightarrow \quad Try\ [f(x)]^{n+1}$$

$\int \frac{x^2}{x^3+1}\,dx$	$\int x\,e^{x^2+1}\,dx$

Quickfire

In your head!

$$\int \frac{4x^3}{x^4-1}\,dx =$$

$$\int \frac{\cos x}{\sin x + 2}\,dx =$$

$$\int \cos x\ e^{\sin x}\,dx =$$

$$\int \cos x\,(\sin x - 5)^7\,dx =$$

$$\int x^2(x^3+5)^7 =$$

Not in your head…

$$\int \frac{x}{(x^2+5)^3}\,dx =$$

Fro Tip: If there's as power around the whole denominator, DON'T use ln: reexpress the expression as a product. e.g. $x(x^2+5)^{-3}$

$\sin^n x \cos x$ vs $\sec^n x \tan x$

Notice when we differentiate $\sin^5 x$, then power decreases:

$$\frac{d}{dx}(\sin^5 x) =$$

However, when we differentiate $\sec^5 x$:

$$\frac{d}{dx}\left((\sec x)^5\right) =$$

Notice that the power of sec didn't go down. Keep this in mind when integrating.

$$\int \sec^4 x \tan x\ dx$$

Test Your Understanding

$$\int \sin x\,(\cos x + 1)^5\,dx$$

$$\int \frac{cosec^2 x}{(2+\cot x)^3}\,dx$$

$$\int \frac{\sec^2 2x}{\tan 2x + 1}\,dx$$

$$\int x(x^2+2)^3\,dx$$

$$\int 5\tan x \sec^2 x\,dx$$

SKILL #5: Integration by Substitution

For some integrations involving a complicated expression, we can make a substitution to turn it into an equivalent integration that is simpler. We wouldn't be able to use 'reverse chain rule' on the following:

Q Use the substitution $u = 2x + 5$ to find $\int x\sqrt{2x+5}\ dx$

The aim is to completely remove any reference to x, and replace it with u. We'll have to work out x and dx so that we can replace them.

STEP 1: Using substitution, work out x and dx (or variant)	
STEP 2: Substitute these into expression.	
STEP 3: Integrate simplified expression.	
STEP 4: Write answer in terms of x.	

How can we tell what substitution to use?

In Edexcel you will usually be given the substitution!
However in some other exam boards, and in STEP, you often aren't.
There's no hard and fast rule, but it's often helpful to replace to replace expressions inside roots, powers or the denominator of a fraction.

	Sensible substitution:
$\int \cos x \sqrt{1+\sin x}\, dx$	$u =$
$\int 6x\, e^{x^2} dx$	$u =$
$\int \frac{xe^x}{1+x} dx$	$u =$
$\int e^{\frac{1-x}{1+x}} dx$	$u =$

Another Example

Q Use the substitution $u = \sin x + 1$ to find $\int \cos x \sin x \,(1 + \sin x)^3 \; dx$

STEP 1: Using substitution, work out x and dx (or variant)	
STEP 2: Substitute these into expression.	
STEP 3: Integrate simplified expression.	
STEP 4: Write answer in terms of x.	

Using substitutions involving implicit differentiation

When a root is involved, it makes thing much tidier if we use $u^2 = \cdots$

Q Use the substitution $u^2 = 2x + 5$ to find $\int x\sqrt{2x+5} \; dx$

STEP 1: Using substitution, work out x and dx (or variant)	
STEP 2: Substitute these into expression.	
STEP 3: Integrate simplified expression.	
STEP 4: Write answer in terms of x.	

This was marginally less tedious than when we used $u = 2x + 5$, as we didn't have fractional powers to deal with.

More examples

Use the substitution $u^2 = x + 1$ to find $\displaystyle\int \frac{x}{(x+1)^{\frac{1}{2}}}\,dx$

Example 4

Use the substitution $x = \frac{2}{3}\tan u$ to find $\displaystyle\int \frac{1}{4+9x^2}\,dx$

Edexcel will usually give you the substitution in the exam question.

However, if you are not provided with a substitution, a 'rule of thumb' is to replace expressions inside roots, powers or the denominator of a fraction by the variable u.

Test Your Understanding

Edexcel C4 Jan 2012 Q6c

(c) Using the substitution $u = 1 + \cos x$, or otherwise, show that

$$\int \frac{2\sin 2x}{(1+\cos x)}\,dx = 4\ln(1+\cos x) - 4\cos x + k,$$

where k is a constant. **(5)**

Hint: You might want to use your double angle formula first.

INTEGRATION BY SUBSTITUTION AND DEFINITE INTEGRALS

When you use integration by substitution to evaluate a definite integral, you do not need to rewrite the expression in terms of x. However, if you use the expression in terms of u, you ***must*** replace the x limits with u limits.

Alternatively, you could convert the integral back to a function of x and use the original limits but this is usually messier!

Example 5

Calculate $\int_0^{\frac{\pi}{2}} \cos x \sqrt{1+\sin x}\; dx$

Example 6

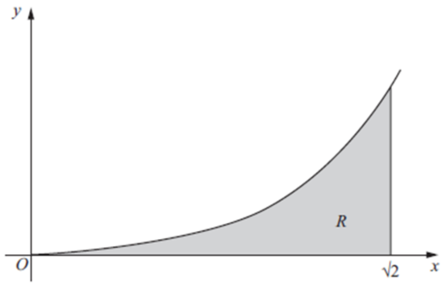


Figure 2 shows a sketch of the curve with equation $y = x^3 \ln(x^2 + 2)$, $x \geq 0$.

The finite region R, shown shaded in Figure 2, is bounded by the curve, the x-axis and the line $x = \sqrt{2}$.

(c) Use the substitution $u = x^2 + 2$ to show that the area of R is

$$\frac{1}{2}\int_2^4 (u-2)\ln u \ \mathrm{d}u .$$

Exercise 11E Page 306

SKILL #6: Integration by Parts

$$\int x \cos x \; dx = ?$$

Just as the Product Rule was used to **differentiate the product** of two expressions, we can often use 'Integration by Parts' to **integrate a product.**

To integrate by parts:

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} \, dx$$

Example 1

$$\int x \cos x \; dx = ?$$

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} \, dx$$

STEP 1: Decide which thing will be u (and which $\frac{dv}{dx}$).

You're about to differentiate u and integrate $\frac{dv}{dx}$, so the idea is to pick them so differentiating u makes it 'simpler', and $\frac{dv}{dx}$ can be integrated easily.

u will always be the x^n term **UNLESS** one term is $\ln x$.

STEP 2: Find $\frac{du}{dx}$ and v.

STEP 3: Use the formula.

I just remember it as "uv minus the integral of the two new things timesed together"

Example 2

Find $\int \boldsymbol{x\,ln\,x\,dx}$

Here, the choice of u must be $ln\,x$ because $ln\,x$ is difficult to integrate

Example 3

Find $\int \boldsymbol{ln\,x\;\,dx}$

Here, the 'trick' is to write the integral as $\int \boldsymbol{1\;\times ln\,x\;\,dx}$

Again, the choice of u must be $ln\,x$

IBP twice! 😐

Q Find $\int x^2 e^x \, dx$

Example 5

Find $\int e^x \cos x \; dx$

Test Your Understanding

Q Find $\int x^2 \sin x \; dx$

SKILL #7: Using Partial Fractions

We saw earlier that we can split some expressions into partial fractions. This allows us to integrate some expressions with more complicated denominators.

Find $\int \frac{2}{x^2-1}\, dx$

Further Examples

Find $\int \frac{x-5}{(x+1)(x-2)}\, dx$

Find $\int \frac{8x^2-19x+1}{(2x+1)(x-2)^2}\, dx$

Test Your Understanding

Edexcel C4 June 2009 Q3

$$f(x) = \frac{4-2x}{(2x+1)(x+1)(x+3)} = \frac{A}{(2x+1)} + \frac{B}{(x+1)} + \frac{C}{(x+3)}.$$

(*a*) Find the values of the constants A, B and C. **(4)**

(*b*) (i) Hence find $\int f(x) \; dx$. **(3)**

(ii) Find $\int_0^2 f(x) \; dx$ in the form $\ln k$, where k is a constant. **(3)**

SKILL #8: Integrating top-heavy algebraic fractions

$$\int \frac{x^2}{x+1}\,dx = ?$$

How would we deal with this? (the clue's in the title)

Test Your Understanding

$$\int \frac{x}{x-1}\,dx$$

$=$

$=$

$$\int \frac{x^3+2}{x+1}\,dx$$

$=$

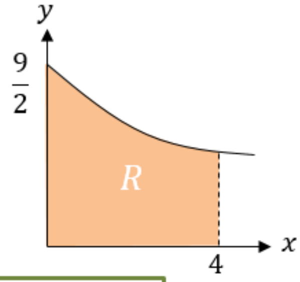
Contrast this with $\int \frac{x-1}{x}\,dx$ which can be integrated more simply:

$$\int \frac{x-1}{x} dx = \int 1 - \frac{1}{x}\,dx = x - \ln|x| + C$$

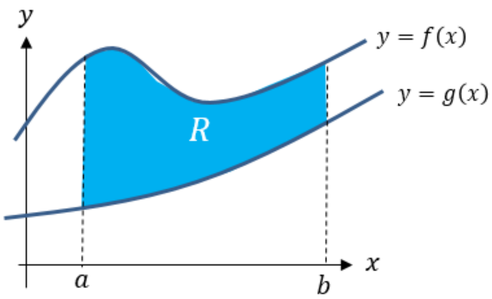
Finding Areas

You're already familiar with the idea that definite integration gives you the (signed) area bound between the curve and the x-axis.
Given your expanded integration skills, you can now find the area under a greater variety of curves.

[Textbook] The diagram shows part of the curve $y = \frac{9}{\sqrt{4+3x}}$
The region R is bounded by the curve, the x-axis and the lines $x = 0$ and $x = 4$, as shown in the diagram. Use integration to find the area of R.



Skill #9: Area between two curves

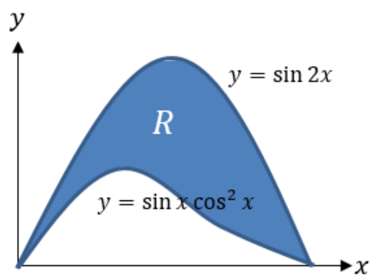


Fro Tip: Ensure you have top curve minus bottom curve.

(This was presented in my Year 1 slides as an 'alternative method')
The areas under the two curves are $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$. It therefore follows the area between them (provided the curves don't overlap) is:

$$R = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$
$$= \int_a^b \big(f(x) - g(x)\big)\,dx$$

[Textbook] The diagram shows part of the curves $y = \sin 2x$ and $y = \sin x \cos^2 x$ where $0 \le x \le \frac{\pi}{2}$. The region R is bounded by the two curves. Use integration to find the area of R.



Test Your Understanding

Edexcel C4 Jan 2009 Q2

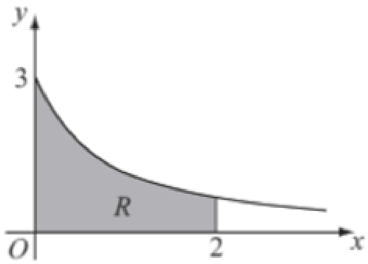
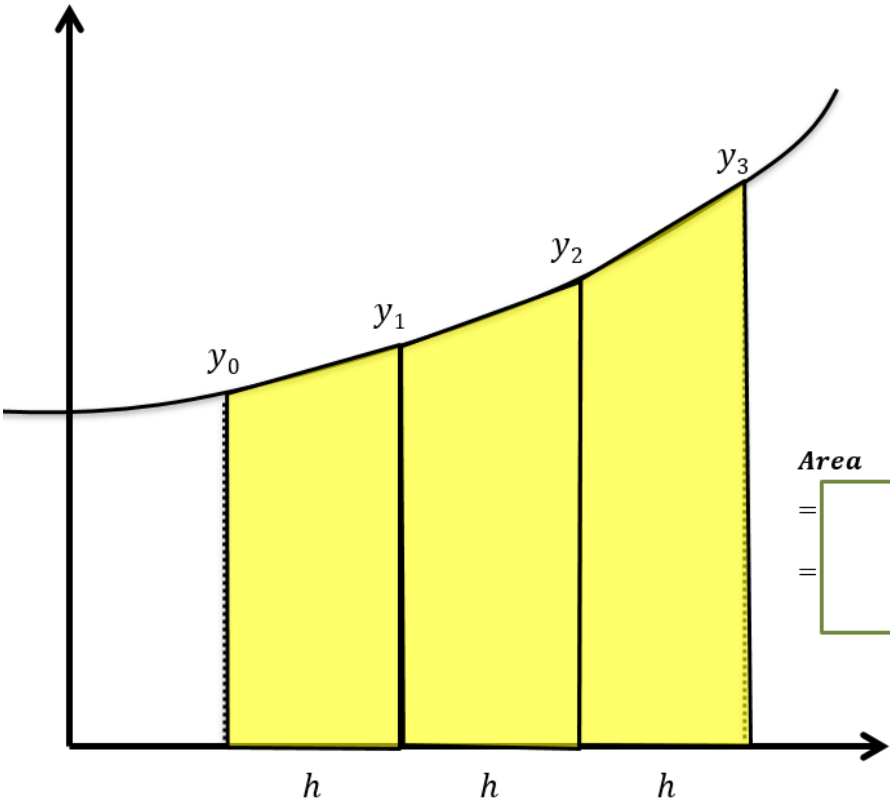


Figure 1

Figure 1 shows part of the curve $y = \dfrac{3}{\sqrt{(1+4x)}}$. The region R is bounded by the curve, the x-axis, and the lines $x = 0$ and $x = 2$, as shown shaded in Figure 1.

(a) Use integration to find the area of R. **(4)**

Skill #10: Trapezium Rule



Sometimes finding the exact area under the graph via integration is difficult. Students who have taken GCSE Maths may be familiar with the idea of **approximating the area under a graph by dividing it into trapeziums of equal width**.

$\boldsymbol{Area}$

$=$

$=$

In general:

width of each trapezium

$$\int_a^b y\,dx \approx \frac{h}{2}\left(y_1 + 2(y_2 + \cdots + y_{n-1}) + y_n\right)$$

Area under curve

is approximately

Example We're approximating the region bounded between $x = 1$, $x = 3$, the x-axis the curve $y = x^2$, using 4 strips.

$\boldsymbol{x}$	
$\boldsymbol{y}$	

Dividing a gap of 2 into 4 strips means each strip will be width 0.5

$h =$

$Area \approx$

Trapezium Rule

Edexcel C2 May 2013 (R) Q2

$$y = \frac{x}{\sqrt{(1+x)}}$$

(*a*) Complete the table below with the value of y corresponding to $x = 1.3$, giving your answer to 4 decimal places.

(1)

Fro Tip: You can generate table with Casio calcs. $Mode \rightarrow 3$ $(Table)$. Use 'Alpha' button to key in X within the function. Press =

x	1	1.1	1.2	1.3	1.4	1.5
y	0.7071	0.7591	0.8090		0.9037	0.9487

(*b*) Use the trapezium rule, with all the values of y in the completed table, to obtain an approximate value for

$$\int_1^{1.5} \frac{x}{\sqrt{(1+x)}}\, \mathrm{d}x$$

giving your answer to 3 decimal places.

You must show clearly each stage of your working.

(4)

Area $\approx$

Further Example

Trapezium Rule: $\int_a^b y\, dx \approx \frac{1}{2}h[y_0 + 2(y_1 + \cdots + y_{n-1}) + y_n]$

Q

Given $I = \int_0^{\frac{\pi}{3}} \sec x\ dx$

a) Find the exact value of I.

b) Use the trapezium rule with two strips to estimate I.

c) Use the trapezium rule with four strips to find a second estimate of I.

d) Find the percentage error in using each estimate.

a

c

b

d

Test Your Understanding

Edexcel C4 June 2014(R) Q2

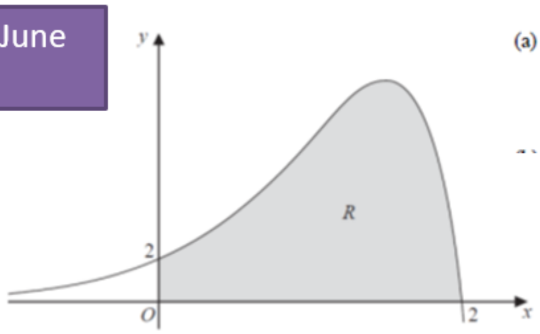


Figure 1

Figure 1 shows a sketch of part of the curve with equation

$$y = (2 - x)e^{2x},$$

The finite region R, shown shaded in Figure 1, is bounded by the curve, the x-axis and the y-axis.

The table below shows corresponding values of x and y for $y = (2 - x)e^{2x}$.

x	0	0.5	1	1.5	2
y	2	4.077	7.389	10.043	0

(*a*) Use the trapezium rule with all the values of y in the table, to obtain an approximation for the area of R, giving your answer to 2 decimal places.

(3)

(*b*) Explain how the trapezium rule can be used to give a more accurate approximation for the area of R.

(1)

(*c*) Use calculus, showing each step in your working, to obtain an exact value for the area of R. Give your answer in its simplest form.

(5)

(a)

(b)

(c)

Integration with Parametric Equations

Suppose we have the following parametric equations:

$$x = t^2$$
$$y = t + 1$$

To find the area under the curve, we want to determine to determine $\int y\, dx$.
The problem however is that y is in terms of t, not in terms of x.

Area: $\int y\, dx = \boxed{}$

Fro Memory Tip: No need to remember the whole new formulae. Just remember that $\frac{dx}{dt} dt = dx$, which follows from the chain rule (and very informally, you can see holds as the dt's cancel)

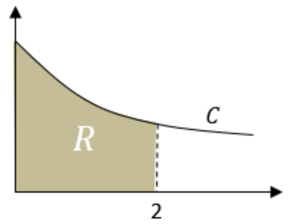
Determine the area bound between the curve with parametric equations $x = t^2$ and $y = t + 1$, the x-axis, and the lines $x = 0$ and $x = 3$.

Further Example

[Textbook] The curve C has parametric equations

$$x = t(1+t), \qquad y = \frac{1}{1+t}, \qquad t \geq 0$$

Find the exact area of the region R, bounded by C, the x-axis and the lines $x = 0$ and $x = 2$.



Test Your Understanding

Edexcel C4 Jan 2013 Q5

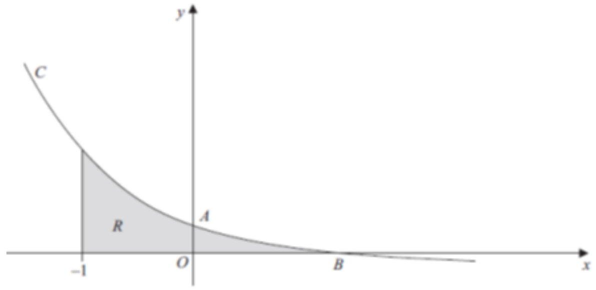


Figure 2 shows a sketch of part of the curve C with parametric equations

$$x = 1 - \frac{1}{2}t, \qquad y = 2^t - 1.$$

The curve crosses the y-axis at the point A and crosses the x-axis at the point B.

(a) Show that A has coordinates (0, 3). **(2)**

(b) Find the x-coordinate of the point B. **(2)**

The region R, as shown shaded in Figure 2, is bounded by the curve C, the line $x = -1$ and the x-axis.

(d) Use integration to find the exact area of R. **(6)**

Helping Hand:

$$\frac{d}{dx}(a^x) = a^x(\ln a)$$

$$\int a^x \; dx = \frac{a^x}{\ln a} + c$$

Exercise ?

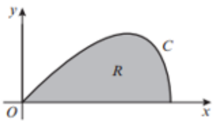
This exercise is not in the current version of the Pearson textbooks as the content was added later. I have temporarily included the exercise subsequently produced by Pearson.

(P) **1** The curve C has parametric equations $x = t^3$, $y = t^2$, $t \geqslant 0$. Show that the exact area of the region bounded by the curve, the x-axis and the lines $x = 0$ and $x = 4$ is $k\sqrt[3]{2}$, where k is a rational constant to be found.

(E/P) **2** The curve C has parametric equations

$$x = \sin t,\ y = \sin 2t,\ 0 \geqslant t \geqslant \frac{\pi}{2}$$

The finite region R is bounded by the curve and the x-axis.
Find the exact area of R. **(6 marks)**



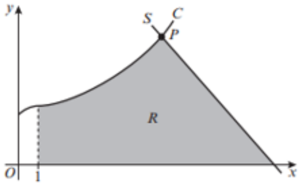
(E/P) **3** This graph shows part of the curve C with parametric equations $x = (t + 1)^2$, $y = \frac{1}{2}t^3 + 3$, $t \geqslant -1$

P is the point on the curve where $t = 2$.
The line S is the normal to C at P.

a Find an equation of S. **(5 marks)**

The shaded region R is bounded by C, S, the x-axis and the line with equation $x = 1$.

b Using integration, find the area of R. **(5 marks)**

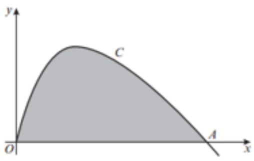


(E/P) **4** The diagram shows the curve C with parametric equations

$$x = 3t^2,\ y = \sin 2t,\ t \geqslant 0.$$

a Write down the value of t at the point A where the curve crosses the x-axis. **(1 mark)**

b Find, in terms of π, the exact area of the shaded region bounded by C and the x-axis. **(6 marks)**

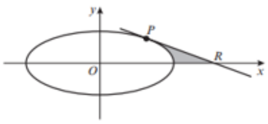


(E/P) **5** The curve shown has parametric equations

$$x = 5\cos\theta,\ y = 4\sin\theta,\ 0 \leqslant \theta < 2\pi$$

a Find the gradient of the curve at the point P at which $\theta = \frac{\pi}{4}$ **(3 marks)**

b Find an equation of the tangent to the curve at the point P. **(3 marks)**

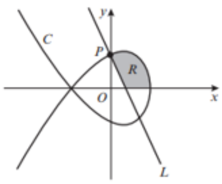


c Find the exact area of the shaded region bounded by the tangent PR, the curve and the x-axis. **(6 marks)**

(E/P) **6** The curve C has parametric equations

$$x = 1 - t^2,\ y = 2t - t^3,\ t \in \mathbb{R}$$

The line L is a normal to the curve at the point P where the curve intersects the positive y-axis. Find the exact area of the region R bounded by the curve C, the line L and the x-axis, as shown on the diagram. **(7 marks)**



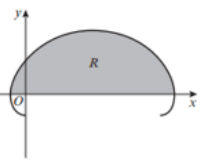
(E/P) **7** The curve shown in the diagram has parametric equations

$$x = t - 2\sin t,\ y = 1 - 2\cos t,\ 0 \leqslant t \leqslant 2\pi$$

a Show that the curve crosses the x-axis where $t = \frac{\pi}{3}$ and $t = \frac{5\pi}{3}$ **(3 marks)**

The finite region R is enclosed by the curve and the x-axis, as shown shaded in the diagram.

b Show that the area R is given by $\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1 - 2\cos t)^2 \, dt$ **(3 marks)**



c Use this integral to find the exact value of the shaded area. **(4 marks)**

ANSWERS

1

2

3

4

5

6

7

SKILL #11: Differential Equations (We're on the home straight!)

Differential equations are equations involving a mix of variables and derivatives, e.g. y, x and $\frac{dy}{dx}$.

'<u>Solving</u>' these equations means to get y in terms of x (with no $\frac{dy}{dx}$).

Q Find the general solution to $\frac{dy}{dx} = xy + y$

Another Example

Q Find the general solution to $(1+x^2)\frac{dy}{dx} = x\tan y$

STEP 1: Get y to the side of $\frac{dy}{dx}$ by dividing and x to the other side.
(you may need to factorise to separate out y first)

STEP 2: Integrate both sides with respect to x.

STEP 2b: If possible, try to combine your constant of integration with other terms
(e.g. by letting $C = \ln k$ where k is another constant)

STEP 3: Make y the subject, if the question asks.

Differential Equations with Boundary Conditions

Q [Textbook] Find the general solution to $\frac{dy}{dx} = -\frac{3(y-2)}{(2x+1)(x+2)}$
Given that $x = 1$ when $y = 4$. Leave your answer in the form $y = f(x)$

Test Your Understanding

Edexcel C4 Jan 2012 Q4

Given that $y = 2$ at $x = \frac{\pi}{4}$, solve the differential equation

$$\frac{dy}{dx} = \frac{3}{y\cos^2 x}.$$

(5)

Key Points on Differential Equations

- Get y on to LHS by dividing (possibly factorising first).
- If after integrating you have ln on the RHS, make your constant of integration $\ln k$.
- Be sure to combine all your ln's together just as you did in C2. E.g.:

$$2\ln|x+1| - \ln|x| \quad \rightarrow$$

- Sub in boundary conditions to work out your constant – better to do sooner rather than later.
- Exam questions ♥ partial fractions combined with differential equations.

Forming differential equations

Differential equations are useful because regularly in real-life, the rate of change of a variable is based on its current value. For example in Year 1, we saw a property of exponential growth is that the **rate of change is proportional to the current value**:

Q The rate of increase of a rabbit population (with population P, where time is t) is **proportional to** the current population.
Form a differential equation, and find its general solution.

Further Example

[Textbook] Water in a manufacturing plant is held in a large cylindrical tank of diameter 20m. Water flows out of the bottom of the tank through a tap at a rate proportional to the cube root of the volume.

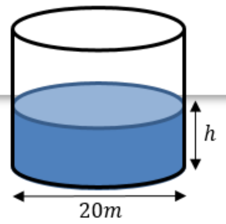
(a) Show that t minutes after the tap is opened, $\frac{dh}{dt} = -k\sqrt[3]{h}$ for some constant k.

(b) Show that the general solution of this differential equation may be written $h = (P - Qt)^{\frac{3}{2}}$, where P and Q are constants.

Initially the height of the water is 27m. 10 minutes later, the height is 8m.

(c) Find the values of the constants P and Q.

(d) Find the time in minutes when the water is at a depth of 1m.



Test Your Understanding

Edexcel C4 June 2005 Q8

Liquid is pouring into a container at a constant rate of 20 $\text{cm}^3\ \text{s}^{-1}$ and is leaking out at a rate proportional to the volume of the liquid already in the container.

(*a*) Explain why, at time t seconds, the volume, $V\ \text{cm}^3$, of liquid in the container satisfies the differential equation

$$\frac{dV}{dt} = 20 - kV,$$

where k is a positive constant.

The container is initially empty.

(*b*) By solving the differential equation, show that

$$V = A + Be^{-kt},$$

giving the values of A and B in terms of k. **(6)**

Given also that $\frac{dV}{dt} = 10$ when $t = 5$,

(*c*) find the volume of liquid in the container at 10 s after the start. **(5)**

(a)

Teachers/Students: I recommend also looking at Edexcel Jan 2008 Q8 which has a part (a) similar to the previous example.

Summary of Functions

$f(x)$	How to deal with it	$\int f(x)dx$ (+constant)	Formula booklet?
$sin\, x$			No
$cos\, x$			No
$tan\, x$			Yes
$sin^2\, x$			No
$cos^2\, x$			No
$tan^2 x$			No
$cosec\, x$			Yes
$sec\, x$			Yes
$cot\, x$			Yes

Summary of Functions

$f(x)$	How to deal with it	$\int f(x)dx$ (+constant)	Formula booklet?
$cosec^2 x$			No!
$sec^2 x$			Yes (but memorise)
$cot^2 x$			No
$sin\, 2x\, cos\, 2x$			No
$\frac{1}{x}$			No
$\ln x$			No
$\frac{x}{x+1}$			
$\frac{1}{x(x+1)}$			

$f(x)$	How to deal with it	$\int f(x)dx$ (+constant)
$\frac{4x}{x^2+1}$		
$\frac{x}{(x^2+1)^2}$		
e^{2x+1} $\frac{1}{1-3x}$		
$x\sqrt{2x+1}$		
$\sin^5 x\ cos\ x$		