Lower 6 Chapter 13

Integration

Course Structure

1. Find y given $\frac{dy}{dx}$
2. Evaluate definite integrals, and hence the area under a curve.
3. Find areas bound between two different lines.

8 Integration	8.1	**Know and use the Fundamental Theorem of Calculus**	**Integration as the reverse process of differentiation. Students should know that for indefinite integrals a constant of integration is required.**
	8.2	**Integrate x^n (excluding $n = -1$) and related sums, differences and constant multiples.**	**For example, the ability to integrate expressions such as $\frac{1}{2}x^2 - 3x^{-\frac{1}{2}}$ and $\frac{(x+2)^2}{x^{\frac{1}{2}}}$ is expected.** x **Given $f'(x)$ and a point on the curve, Students should be able to find an equation of the curve in the form $y = f(x)$.**
8 Integration *continued*	8.3	**Evaluate definite integrals; use a definite integral to find the area under a curve** and the area between two curves	Students will be expected to be able to evaluate the area of a region bounded by a curve and given straight lines, or between two curves. This includes curves defined parametrically. For example, find the finite area bounded by the curve $y = 6x - x^2$ and the line $y = 2x$ Or find the finite area bounded by the curve $y = x^2 - 5x + 6$ and the curve $y = 4 - x^2$.

<u>Integrating x^n terms</u>

Integration is the **opposite of differentiation.**

Consider:

If $\frac{dy}{dx} = 3x^2$, what could $f(x)$?

Examples

Find y when:

1. $\frac{dy}{dx} = 4x^3$
2. $\frac{dy}{dx} = x^5$
3. $\frac{dy}{dx} = 3x^{\frac{1}{2}}$
4. $\frac{dy}{dx} = \frac{4}{\sqrt{x}}$

5. $\frac{dy}{dx} = 5x^{-2}$

6. $\frac{dy}{dx} = 4x^{\frac{2}{3}}$

7. $\frac{dy}{dx} = 10x^{-\frac{2}{7}}$

Test Your Understanding

Find $f(x)$ when:

$f'(x) = 2x + 7$

$f'(x) = x^2 - 1$

$f'(x) = \frac{2}{x^7}$

$f'(x) = \sqrt[3]{x} =$

$f'(x) = 33x^{\frac{5}{6}}$

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Integration Notation

The following notation could be used to differentiate an expression:

The dx here means differentiating "with respect to x".

$$\frac{d}{dx}(5x^2) = 10x$$

There is similarly notation for integrating an expression:

$$\int 10x \; dx = 5x^2 + c$$

"Integrate…"

"…this expression"

"…with respect to x"
(the dx is needed just as it was needed in the differentiation notation at the top of this slide)

This is known as **indefinite integration**, in contrast to definite integration, which we'll see later in the chapter.

It is called 'indefinite' because the exact expression is unknown (due to the $+c$).

Examples

1. Find $\int (x^{-\frac{3}{2}} + 2) \; dx$

2. Find $\int (6t^2 - 1) \; dt$

3. Find $\int (px^3 + q) \; dx$ where p and q are constants.

Test Your Understanding

Given that $y = 2x^5 + \frac{6}{\sqrt{x}}$, $x > 0$, find in their simplest form

(b) $\int y \, \mathrm{d}x$ **(3)**

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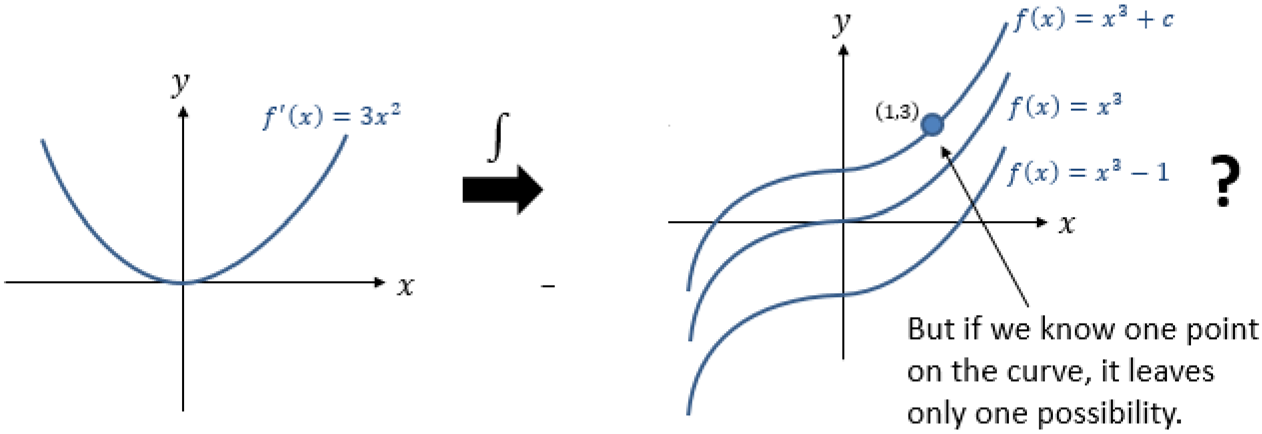
Finding the Constant of Integration

Recall that when we integrate, we get a constant of integration, which could be any real value. This means **we don't know what the exact original function was**.



Example

The curve with equation $y = f(x)$ passes through $(1,3)$. Given that $f'(x) = 3x^2$, find the equation of the curve.

Test Your Understanding

A curve with equation $y = f(x)$ passes through the point (4, 25).

Given that

$$f'(x) = \frac{3}{8}x^2 - 10x^{-\frac{1}{2}} + 1, \qquad x > 0$$

(a) find f(x), simplifying each term.

(5)

(b) Find an equation of the normal to the curve at the point (4, 25).

Give your answer in the form $ax + by + c = 0$, where a, b and c are integers to be found.

(5)

Ex 13C pg 294

Definite Integral

The most useful use of integration is that **it finds the area under a graph**. Before we do this, we need to understand how to find a **definite integral**.

Examples

1. $f(x) = 4x^3$

2. $\int_{-3}^{3} x^2 + 1\, dx =$

3. Given that P is a constant and $\int_1^5 (2Px + 7)\, dx = 4P^2$, show that there are two possible values for P and find these values.

Extension

1. *[MAT 2009 1A]* The smallest value of

$$I(a) = \int_0^1 (x^2 - a)^2 \, dx$$

as a varies, is what?

2. *[MAT 2015 1D]* Let

$f(x) = \int_0^1 (xt)^2 \, dt$ and $g(x) = \int_0^x t^2 \, dt$

Let $A > 0$. Which of the following statements are true?

A) $g\big(f(A)\big)$ is always bigger than $f\big(g(A)\big)$

B) $f\big(g(A)\big)$ is always bigger than $g\big(f(A)\big)$

C) They are always equal.

D) $f\big(g(A)\big)$ is bigger if $A < 1$, and $g\big(f(A)\big)$ is bigger if $A > 1$.

E) $g\big(f(A)\big)$ is bigger if $A < 1$, and $f\big(g(A)\big)$ is bigger if $A > 1$.

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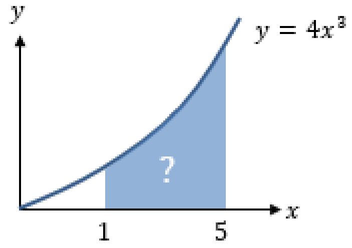
Areas Under Curves

Consider our previous example $\int_1^5 4x^3\ dx$. This definite integral gives the area bounded by the curve and the lines x = 1 and x = 5.



> The definite integral $\int_{\boldsymbol{b}}^{\boldsymbol{a}} \boldsymbol{f(x)\ dx}$ gives the **area** between a positive curve $y = f(x)$, the $\boldsymbol{x}$**-axis**, and the lines $x = a$ and $x = b$.

Example

Find the area of the finite region between the curve with equation $y = 20 - x - x^2$ and the x-axis.

<u>Test Your Understanding</u>

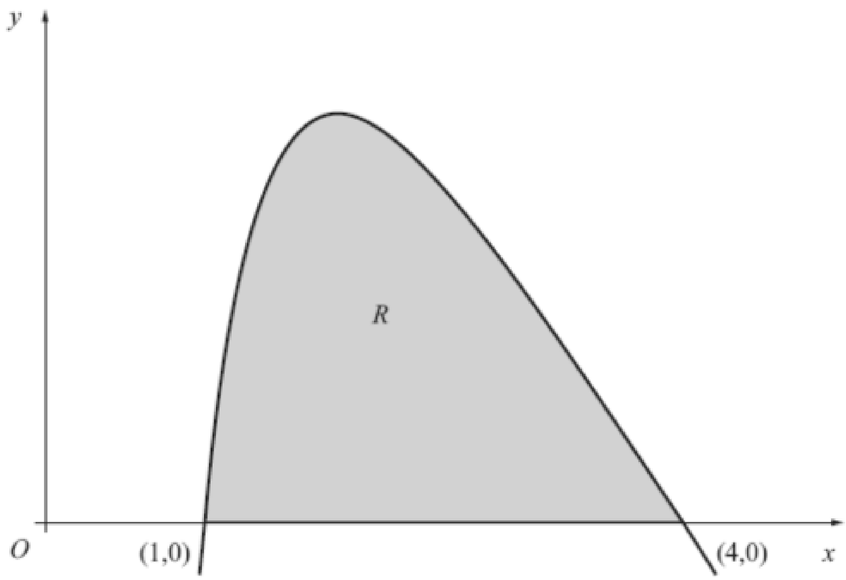


Figure 2

The finite region R, as shown in Figure 2, is bounded by the x-axis and the curve with equation

$$y = 27 - 2x - 9\sqrt{x} - \frac{16}{x^2}, \qquad x > 0.$$

The curve crosses the x-axis at the points (1, 0) and (4, 0).

(c) Use integration to find the exact value for the area of R.

(6)

Extension

[MAT 2007 1H] Given a function $f(x)$, you are told that

$$\int_0^1 3f(x)\,dx + \int_1^2 2f(x)\,dx = 7 \qquad \int_0^2 f(x)\,dx + \int_1^2 f(x)\,dx = 1$$

It follows that $\int_0^2 f(x)\ dx$ equals what?

[MAT 2011 1G]
A graph of the function $y = f(x)$ is sketched on the axes below:
What is the value of $\int_{-1}^{1} f(x^2 - 1)\ dx$?

Ex 13E pg 299

Negative Areas

Sketch the curve $y = x(x-1)(x-2)$.

Now calculate $\int_0^2 x(x-1)(x-2)\ dx$.

Why is this result surprising?

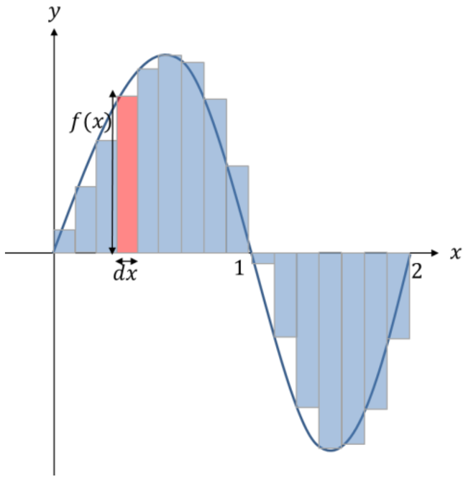
Integration $\int f(x)\,dx$ is just the sum of areas of infinitely thin rectangles, where the current y value (i.e. $f(x)$) is each height, and the widths are dx.

i.e. The area of each is $f(x) \times dx$

The problem is, when $f(x)$ is negative, then $f(x) \times dx$ is negative, i.e. a negative area!

The result is that the 'positive area' from 0 to 1 is cancelled out by the 'negative area' from 1 to 2, giving an overall 'area' of 0.

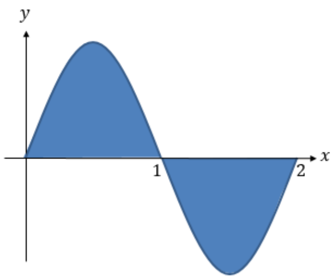
So how do we resolve this?



This explains the dx in the $\int f(x)\,dx$, which effectively means "the sum of the areas of strips, each of area $f(x) \times dx$. So the dx is not just part of the $\int$ notation, it's behaving as a physical quantity! (i.e. length

Example

Find the total area bound between the curve $y = x(x-1)(x-2)$ and the x-axis.



Test Your Understanding

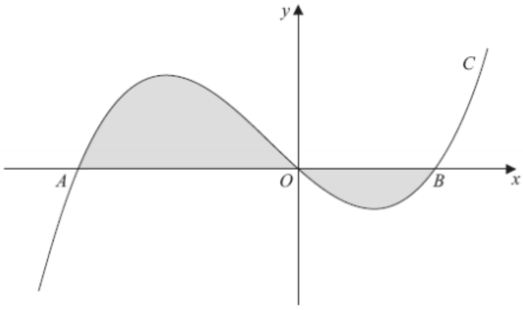


Figure 3

Figure 3 shows a sketch of part of the curve C with equation

$$y = x(x + 4)(x - 2).$$

The curve C crosses the x-axis at the origin O and at the points A and B.

(a) Write down the x-coordinates of the points A and B.

(1)

The finite region, shown shaded in Figure 3, is bounded by the curve C and the x-axis.

(b) Use integration to find the total area of the finite region shown shaded in Figure 3.

(7)

Extension

[MAT 2010 1I] For a positive number a, let

$$I(a) = \int_0^a \left(4 - 2^{x^2}\right) dx$$

Then $\frac{dI}{da} = 0$ when a is what value?

[STEP I 2014 Q3]

The numbers a and b, where $b > a \geq 0$, are such that

$$\int_a^b x^2\, dx = \left(\int_a^b x\, dx\right)^2$$

(i) In the case $a = 0$ and $b > 0$, find the value of b.

(ii) In the case $a = 1$, show that b satisfies

$$3b^3 - b^2 - 7b - 7 = 0$$

Show further, with the help of a sketch, that there is only one (real) value of b that satisfies the equation and that it lies between 2 and 3.

(iii) Show that $3p^2 + q^2 = 3p^2 q$, where $p = b + a$ and $q = b - a$, and express p^2 in terms of q. Deduce that $1 < b - a \leq \frac{4}{3}$

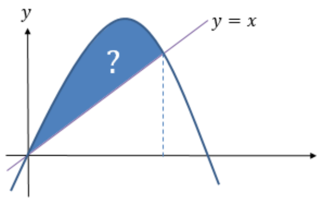
Ex 13F pg 301

Areas Between Curves and Lines

We are often interested in areas formed between curves and lines. It is important to sketch the graph to consider which areas we need to calculate.

Example

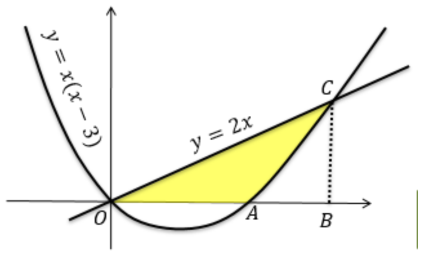
Determine the area between the lines with equations $y = x(4 - x)$ and $y = x$



Example

The diagram shows a sketch of the curve with equation $y = x(x - 3)$ and the line with equation $y = 2x$.

Find the area of the shaded region OAC.



Test Your Understanding

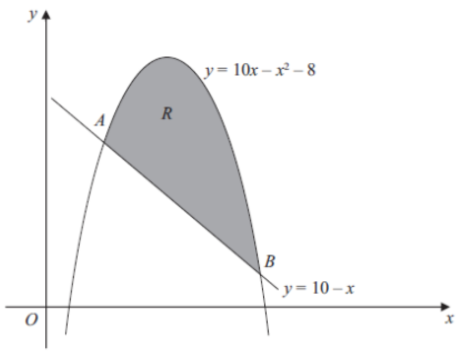


Figure 2 shows the line with equation $y = 10 - x$ and the curve with equation $y = 10x - x^2 - 8$.

The line and the curve intersect at the points A and B, and O is the origin.

(a) Calculate the coordinates of A and the coordinates of B.

(5)

The shaded area R is bounded by the line and the curve, as shown in Figure 2.

(b) Calculate the exact area of R.

(7)

Alternative Method:
If the top curve has equation $y = f(x)$ and the bottom curve $y = g(x)$, the area between them is:

$$\int_b^a \big(f(x) - g(x)\big)\,dx$$

This means you can integrate a single expression to get the final area, without any adjustment required after.

Extension

[MAT 2005 1A] What is the area of the region bounded by the curves $y = x^2$ and $y = x + 2$?

[MAT 2016 1H] Consider two functions

$$f(x) = a - x^2 g(x) = x^4 - a$$

For precisely which values of $a > 0$ is the area of the region bounded by the x-axis and the curve $y = f(x)$ bigger than the area of the region bounded by the x-axis and the curve $y = g(x)$?

(Your answer should be an inequality in terms of a)

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