Negative Areas

Sketch the curve $y = x(x-1)(x-2)$.

Now calculate $\int_0^2 x(x-1)(x-2)\ dx$.

Why is this result surprising?

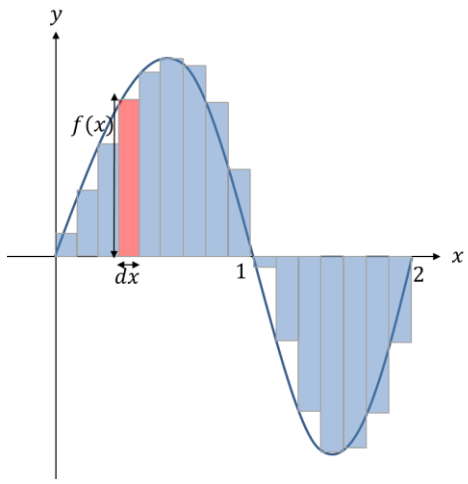
Integration $\int f(x)\, dx$ is just the sum of areas of infinitely thin rectangles, where the current y value (i.e. $f(x)$) is each height, and the widths are dx.

i.e. The area of each is $f(x) \times dx$

The problem is, when $f(x)$ is negative, then $f(x) \times dx$ is negative, i.e. a negative area!

The result is that the 'positive area' from 0 to 1 is cancelled out by the 'negative area' from 1 to 2, giving an overall 'area' of 0.

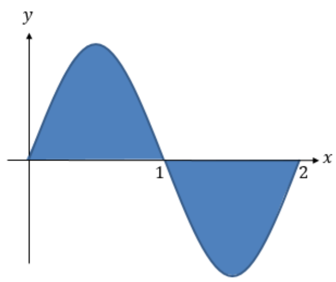
So how do we resolve this?



This explains the dx in the $\int f(x)\, dx$, which effectively means "the sum of the areas of strips, each of area $f(x) \times dx$. So the dx is not just part of the $\int$ notation, it's behaving as a physical quantity! (i.e. length

Example

Find the total area bound between the curve $y = x(x-1)(x-2)$ and the x-axis.



Test Your Understanding

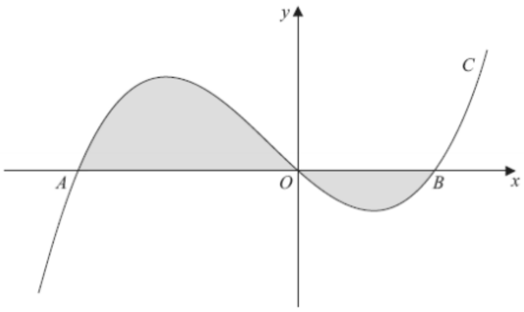


Figure 3

Figure 3 shows a sketch of part of the curve C with equation

$$y = x(x+4)(x-2).$$

The curve C crosses the x-axis at the origin O and at the points A and B.

(*a*) Write down the x-coordinates of the points A and B.

(1)

The finite region, shown shaded in Figure 3, is bounded by the curve C and the x-axis.

(*b*) Use integration to find the total area of the finite region shown shaded in Figure 3.

(7)

Extension

[MAT 2010 1I] For a positive number a, let

$$I(a) = \int_0^a \left(4 - 2^{x^2}\right) dx$$

Then $\frac{dI}{da} = 0$ when a is what value?

[STEP I 2014 Q3]

The numbers a and b, where $b > a \geq 0$, are such that

$$\int_a^b x^2\, dx = \left(\int_a^b x\, dx\right)^2$$

(i) In the case $a = 0$ and $b > 0$, find the value of b.

(ii) In the case $a = 1$, show that b satisfies

$$3b^3 - b^2 - 7b - 7 = 0$$

Show further, with the help of a sketch, that there is only one (real) value of b that satisfies the equation and that it lies between 2 and 3.

(iii) Show that $3p^2 + q^2 = 3p^2 q$, where $p = b + a$ and $q = b - a$, and express p^2 in terms of q. Deduce that $1 < b - a \leq \frac{4}{3}$

Ex 13F pg 301