Integration with Parametric Equations

Suppose we have the following parametric equations:

$$x = t^2$$
$$y = t + 1$$

To find the area under the curve, we want to determine to determine $\int y\, dx$.
The problem however is that y is in terms of t, not in terms of x.

Area: $\int y\, dx =$ [ ]

Fro Memory Tip: No need to remember the whole new formulae. Just remember that $\frac{dx}{dt}dt = dx$, which follows from the chain rule (and very informally, you can see holds as the dt's cancel)

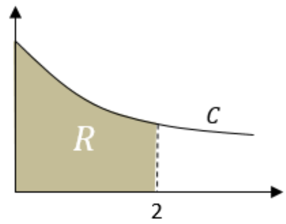
Determine the area bound between the curve with parametric equations $x = t^2$ and $y = t + 1$, the x-axis, and the lines $x = 0$ and $x = 3$.

Further Example

[Textbook] The curve C has parametric equations

$$x = t(1+t), \qquad y = \frac{1}{1+t}, \qquad t \geq 0$$

Find the exact area of the region R, bounded by C, the x-axis and the lines $x = 0$ and $x = 2$.



Test Your Understanding

Edexcel C4 Jan 2013 Q5

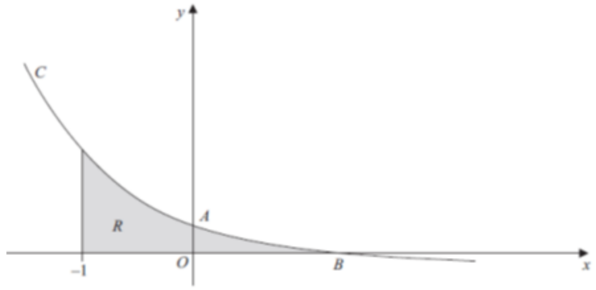


Figure 2 shows a sketch of part of the curve C with parametric equations

$$x = 1 - \frac{1}{2}t, \qquad y = 2^t - 1.$$

The curve crosses the y-axis at the point A and crosses the x-axis at the point B.

(a) Show that A has coordinates (0, 3). **(2)**

(b) Find the x-coordinate of the point B. **(2)**

The region R, as shown shaded in Figure 2, is bounded by the curve C, the line $x = -1$ and the x-axis.

(d) Use integration to find the exact area of R. **(6)**

Helping Hand:

$$\frac{d}{dx}(a^x) = a^x(\ln a)$$

$$\int a^x \, dx = \frac{a^x}{\ln a} + c$$

Exercise ?

This exercise is not in the current version of the Pearson textbooks as the content was added later. I have temporarily included the exercise subsequently produced by Pearson.

ANSWERS

1

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4

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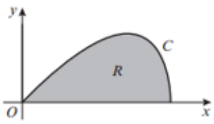
7

(P) **1** The curve C has parametric equations $x = t^3$, $y = t^2$, $t \geqslant 0$. Show that the exact area of the region bounded by the curve, the x-axis and the lines $x = 0$ and $x = 4$ is $k\sqrt[3]{2}$, where k is a rational constant to be found.

(E/P) **2** The curve C has parametric equations

$$x = \sin t,\ y = \sin 2t,\ 0 \geqslant t \geqslant \frac{\pi}{2}$$

The finite region R is bounded by the curve and the x-axis.
Find the exact area of R. **(6 marks)**

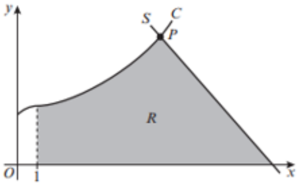


(E/P) **3** This graph shows part of the curve C with parametric equations $x = (t + 1)^2$, $y = \frac{1}{2}t^3 + 3$, $t \geqslant -1$
P is the point on the curve where $t = 2$.
The line S is the normal to C at P.

a Find an equation of S. **(5 marks)**

The shaded region R is bounded by C, S, the x-axis and the line with equation $x = 1$.

b Using integration, find the area of R. **(5 marks)**

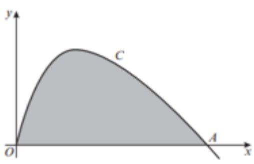


(E/P) **4** The diagram shows the curve C with parametric equations

$$x = 3t^2,\ y = \sin 2t,\ t \geqslant 0.$$

a Write down the value of t at the point A where the curve crosses the x-axis. **(1 mark)**

b Find, in terms of π, the exact area of the shaded region bounded by C and the x-axis. **(6 marks)**



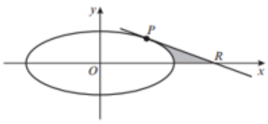
(E/P) **5** The curve shown has parametric equations

$$x = 5\cos\theta,\ y = 4\sin\theta,\ 0 \leqslant \theta < 2\pi$$

a Find the gradient of the curve at the point P at which $\theta = \frac{\pi}{4}$ **(3 marks)**

b Find an equation of the tangent to the curve at the point P. **(3 marks)**

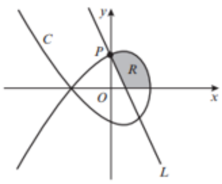
c Find the exact area of the shaded region bounded by the tangent PR, the curve and the x-axis. **(6 marks)**



(E/P) **6** The curve C has parametric equations

$$x = 1 - t^2,\ y = 2t - t^3,\ t \in \mathbb{R}$$

The line L is a normal to the curve at the point P where the curve intersects the positive y-axis. Find the exact area of the region R bounded by the curve C, the line L and the x-axis, as shown on the diagram. **(7 marks)**



(E/P) **7** The curve shown in the diagram has parametric equations

$$x = t - 2\sin t,\ y = 1 - 2\cos t,\ 0 \leqslant t \leqslant 2\pi$$

a Show that the curve crosses the x-axis where $t = \frac{\pi}{3}$ and $t = \frac{5\pi}{3}$ **(3 marks)**

The finite region R is enclosed by the curve and the x-axis, as shown shaded in the diagram.

b Show that the area R is given by $\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1 - 2\cos t)^2\, dt$ **(3 marks)**

c Use this integral to find the exact value of the shaded area. **(4 marks)**