SKILL #5: Integration by Substitution

For some integrations involving a complicated expression, we can make a substitution to turn it into an equivalent integration that is simpler. We wouldn't be able to use 'reverse chain rule' on the following:

Q Use the substitution $u = 2x + 5$ to find $\int x\sqrt{2x+5}\ dx$

The aim is to completely remove any reference to x, and replace it with u. We'll have to work out x and dx so that we can replace them.

STEP 1: Using substitution, work out x and dx (or variant)	
STEP 2: Substitute these into expression.	
STEP 3: Integrate simplified expression.	
STEP 4: Write answer in terms of x.	

How can we tell what substitution to use?

In Edexcel you will usually be given the substitution!
However in some other exam boards, and in STEP, you often aren't.
There's no hard and fast rule, but it's often helpful to replace to replace expressions inside roots, powers or the denominator of a fraction.

	Sensible substitution:
$\int \cos x\sqrt{1+\sin x}\,dx$	$u =$
$\int 6x\, e^{x^2} dx$	$u =$
$\int \frac{xe^x}{1+x} dx$	$u =$
$\int e^{\frac{1-x}{1+x}} dx$	$u =$

Another Example

Q Use the substitution $u = \sin x + 1$ to find $\int \cos x \sin x \, (1 + \sin x)^3 \; dx$

STEP 1: Using substitution, work out x and dx (or variant)	
STEP 2: Substitute these into expression.	
STEP 3: Integrate simplified expression.	
STEP 4: Write answer in terms of x.	

Using substitutions involving implicit differentiation

When a root is involved, it makes thing much tidier if we use $u^2 = \cdots$

Q Use the substitution $u^2 = 2x + 5$ to find $\int x\sqrt{2x+5} \; dx$

STEP 1: Using substitution, work out x and dx (or variant)	
STEP 2: Substitute these into expression.	
STEP 3: Integrate simplified expression.	
STEP 4: Write answer in terms of x.	

This was marginally less tedious than when we used $u = 2x + 5$, as we didn't have fractional powers to deal with.

More examples

Use the substitution $u^2 = x + 1$ to find $$\int \frac{x}{(x+1)^{\frac{1}{2}}}\,dx$$

Example 4

Use the substitution $x = \frac{2}{3}\tan u$ to find $$\int \frac{1}{4+9x^2}\,dx$$

Edexcel will usually give you the substitution in the exam question.

However, if you are not provided with a substitution, a 'rule of thumb' is to replace expressions inside roots, powers or the denominator of a fraction by the variable u.

Test Your Understanding

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(c) Using the substitution $u = 1 + \cos x$, or otherwise, show that

$$\int \frac{2\sin 2x}{(1+\cos x)}\,dx = 4\ln(1+\cos x) - 4\cos x + k,$$

where k is a constant. **(5)**

Hint: You might want to use your double angle formula first.

INTEGRATION BY SUBSTITUTION AND DEFINITE INTEGRALS

When you use integration by substitution to evaluate a definite integral, you do not need to rewrite the expression in terms of x. However, if you use the expression in terms of u, you ***must*** replace the x limits with u limits.

Alternatively, you could convert the integral back to a function of x and use the original limits but this is usually messier!

Example 5

Calculate $\int_0^{\frac{\pi}{2}} \cos x\sqrt{1+\sin x}\;dx$

Example 6

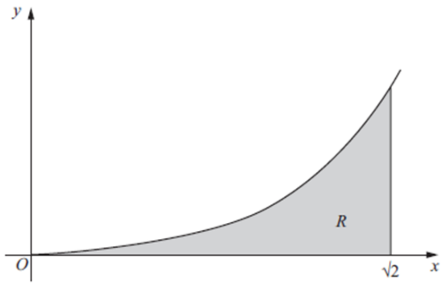

Figure 2 shows a sketch of the curve with equation $y = x^3 \ln (x^2 + 2)$, $x \geq 0$.

The finite region R, shown shaded in Figure 2, is bounded by the curve, the x-axis and the line $x = \sqrt{2}$.

(c) Use the substitution $u = x^2 + 2$ to show that the area of R is

$$\frac{1}{2}\int_2^4 (u-2)\ln u \; \mathrm{d}u.$$

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